D-MATH | Complex Analysis | HS 2022
Prof. Emmanuel Kowalski

Exercise sheet 5

Exercise worth bonus points: Exercise 1

1. For $r > 0$, let U_r be the open set in $\mathbf{C}$ consisting of complex numbers z such that $\mathrm{Re}(z) > r$ and $-z$ is not a non-negative integer (so $z = 0, -1, -2, \ldots$ are not in any U_r). Let U be the open set of those $z \in \mathbf{C}$ such that $-z$ is not a non-negative integer (so all z except $0, -1, \ldots,$ are in U).

   (a) Show that the function defined by the improper Riemann integral
   $$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt$$
   is well-defined for $z \in U_1$. (Here $t^{z-1} = \exp((z-1)\log t)$ for $t > 0$ and is defined to be 0 for $t = 0$.)

   (b) Show that $\Gamma \in \mathcal{H}(U_1)$. (Hint: first prove that for each integer $n \geqslant 1$, the formula
   $$\Gamma_n(z) = \int_0^{n} e^{-t} t^{z-1} dt$$
   defines a function holomorphic in U_1, and then prove that for any real number $A > 0$, the sequence (Γ_n) converges to Γ uniformly for z such that $1 \leqslant \mathrm{Re}(z) \leqslant A$.)

   (c) Show that $\Gamma(z+1) = z\Gamma(z)$ for $z \in U_1$.

   (d) Deduce that there exists a unique holomorphic function on U_0 which coincides with Γ on U_1.

   (e) Deduce further that there exists a unique holomorphic function on U which coincides with Γ on U_1.

   (f) Explain how one could compute $\Gamma(-2+3i)$ by reducing to approximating an integral.

2. Show that the following functions exist and are holomorphic on the indicated open

sets; furthermore, give a similar expression for their derivatives:

$$f_1(z) = \sum_{n=1}^{\infty} \frac{z^n}{1-z^n} \quad \text{on } D_1(0)$$
$$f_2(z) = \int_0^1 (1-tz)^4 e^{tz} dt \quad \text{on } \mathbf{C}$$
$$f_3(z) = \sum_{n=0}^{+\infty} n^2 \exp(2i\pi n^3 z) \quad \text{on } \{z \in \mathbf{C} \mid \operatorname{Im}(z) > 0\}.$$

3. Let $g\colon [0, 2\pi] \to \mathbf{C}$ be a continuous function such that $g(0) = g(2\pi)$. Let γ be the unit circle in the $\mathbf{C}$ with the counterclockwise orientation. Define a function $\widetilde{g}$ on γ by
$$\widetilde{g}(e^{i\theta}) = g(\theta)$$
for $\theta \in [0, 2\pi]$.

   (a) Show that the line integral
   $$\frac{1}{2i\pi} \int_\gamma \frac{\widetilde{g}(w)}{w-z} dw$$
   exists for all $z \in D_1(0)$.

   (b) Show that
   $$f(z) = \frac{1}{2i\pi} \int_\gamma \frac{\widetilde{g}(w)}{w-z} dw$$
   defines a holomorphic function on $D_1(0)$.

   (c) Compute the derivative of f as another line integral along γ.

4. (a) Let $f(z) := \sin(z^2)$. Find the zeros $z_0 \in \mathbb{C}$ of f and their orders.

   (b) Let $p(z) := 1 + a_1 z + \ldots + a_n z^n$ be a polynomial. Find the order of the zero $z_0 = 0$ of the function $f(z) := e^z - p(z)$ depending on the polynomial $p(z)$.