Solutions - ES5

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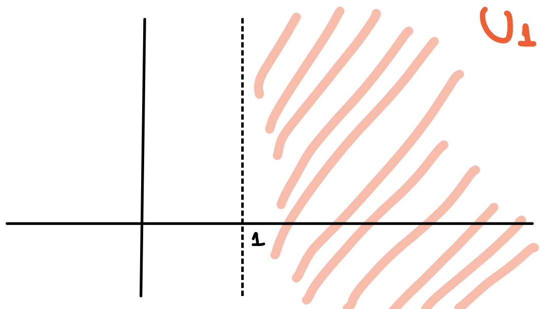


a) Let z be such that $Re(z) > 1$. We observe that

$$\int_0^\infty \left| e^{-t} \cdot e^{(z-1)\log t} \right| dt = \int_0^\infty e^{-t} \cdot e^{(Re(z)-1)\cdot \log t} \, dt .$$

To show that the last integral is well defined

we observe that since $\lim_{t \to \infty} \frac{t}{\log t} = \infty$ there exists

$c > 1$ that depends on $Re(z)$ such that

$$-t + \log(t) \cdot (Re\, z - 1) \leq -t/2 \quad \forall \quad t \in [c, \infty).$$

Thus,

$$\int_0^\infty e^{-t} \cdot e^{(Re(z)-1)\cdot \log t} \, dt = \int_0^c e^{-t} \cdot e^{(Re\, z-1)\log t} \, dt + \int_c^\infty e^{-t} \cdot e^{(Re\, z-1)\log t} \, dt$$

and

observe that $\operatorname{Re} z - 1 > 0$

$$\int_0^c e^{-t} \cdot e^{(\operatorname{Re} z - 1)\log t}\, dt \leq e^{(\operatorname{Re} z - 1)\cdot \log c} \int_0^c e^{-t}\, dt < +\infty$$

$$\int_c^\infty e^{-t} \cdot e^{(\operatorname{Re} z - 1)\log t}\, dt \leq \int_c^\infty e^{-t/2}\, dt < +\infty$$, so the function

is well defined.

b) To show that Γ_m is holomorphic we use Morera's Thm.

I. Γ_m is continuous.

Let $z_k \longrightarrow z$, $z_k, z \in U_1$. Observe that $[0,m]$ is compact and the function being integrated is continuous so,

$$\lim_{k\to\infty} \int_0^m e^{-t} \cdot t^{z_k - 1}\, dt = \int_0^m \lim_{k\to\infty} e^{-t} \cdot t^{z_k - 1}\, dt = \Gamma_m(z).$$

II. Integrating in triangles.

Let Δ be any triangle in U_1.

$$\int_\Delta \Gamma_m(z)\, dz = \int_\Delta \int_0^m e^{-t} \cdot t^{z-1}\, dz = \int_0^m e^{-t} \int_\Delta t^{z-1}\, dz .$$

Since we are integrating in compacts a continuous function we can use Fubini's Thm above. To conclude we observe that t^{z-1} is holomorphic in U_1, so by Cauchy's Thm we have:

$$\int_{\Delta} \Gamma_m(z)dz = \int_0^m e^{-t} \int_{\Delta} t^{z-1} dz = 0 .$$

So Γ_m is holomorphic by Morera's Thm.

Now let $A > 1$ and $m \geq 1$. Let $z \in \{w : 1 \leq Re(w) \leq A\}$. Observe that

$$|\Gamma(z) - \Gamma_m(z)| \leq \int_m^\infty e^{-t} \cdot e^{\log t\,(A-1)} dt, \quad \text{since } m \geq 1$$

and thus $\log t \geq 0$ and $Re(z) - 1 \leq A-1$.

Since we have $\int_1^\infty e^{-t} \cdot t^{A-1} dt < +\infty$ then

$$\lim_{m\to\infty} |\Gamma(z) - \Gamma(z)| \leq \lim_{m\to\infty} \int_m^\infty e^{-t} \cdot e^{\log t\,(A-1)} dt = 0.$$

Observe that the limit is uniform in the region $\{w : 1 \leq Re(w) \leq A\}$. So we conclude that Γ is also a holomorphic function.

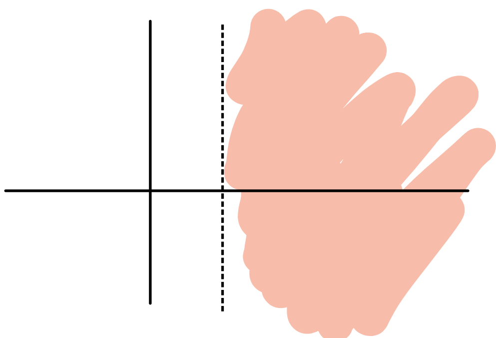
c) $\Gamma(z+1) = z\Gamma(z) \quad \forall\, z \in U_1$.

To show the identity we integrate by parts:

$$\Gamma(z+1) = \int_0^\infty e^{-t} \cdot t^z \, dz = \underbrace{-t^z \cdot e^{-t} \Big|_0^\infty}_{=0} - \int_0^\infty (-e^{-t}) \cdot z \cdot t^{z-1} \, dt$$

$$= z \int_0^\infty e^{-t} \cdot t^{z-1} \, dt = z\Gamma(z).$$

d) Recall that $\Gamma(z)$ is defined in U_1.



We can define a function $\tilde{\Gamma}_0$ in the strip $\{z : 0 < Re(z)\}$ by

$$\tilde{\Gamma}_0(z) = \frac{\Gamma(z+1)}{z}.$$

Observe that $\tilde{\Gamma}_0$ is well-defined and it is holomorphic for $Re(z) > 0$. Since $\tilde{\Gamma}_0$ coincides with Γ for $Re\, z > 1$ from the previous item we know that $\tilde{\Gamma}_0$ is an analytic continuation of Γ.

e) We proceed by defining $\tilde{\Gamma}_1$ in $\{z : -1/2 < Re\, z\} \setminus \{0\}$

$$\tilde{\Gamma}_1(z) = \frac{\Gamma(z+1)}{z}.$$

By the previous argument $\tilde{\Gamma}_1$ is an analytic continuation of Γ in $U_{-1/2}$. Inductively, we conclude that Γ extends to an analytic function in U.

f) Observe that

$$\Gamma(-2+3i) = \frac{\Gamma(-1+3i)}{(-2+3i)} = \frac{\Gamma(3i)}{(-2+3i)(-1+3i)} = \frac{\Gamma(1+3i)}{(-2+3i)(-1+3i)\,3i}$$

$$= \frac{\Gamma(2+3i)}{(-2+3i)(-1+3i)\,3i\,(1+3i)}$$

and $\Gamma(2+3i)$ is defined in terms of the integral

$$\Gamma(2+3i) = \int_0^\infty e^{-t} \cdot t^{1+3i}\, dt.$$

2) $f_1(z) = \sum_{m=0}^{\infty} \frac{z^m}{1-z^m}$ on $D_1(0)$.

Observe that $\forall\, m \geq 1$, $\frac{z^m}{1-z^m}$ is holomorphic in $D_1(0)$,

thus,

$$f_{1,N}(z) = \sum_{m=0}^{N} \frac{z^m}{1-z^m} \quad \text{is also holomorphic } \forall\, N \geq 1.$$

We show that $f_{1,N} \longrightarrow f_1$ uniformly in compacts of $B_1(0)$. Let $K \subset B_1(0)$ be a compact and define

$r = \sup_{z \in K} |z| < 1$. Then, $\forall\, z \in K$

$$\left| f_1(z) - f_{1,N}(z) \right| \leq \sum_{m=N}^{\infty} \frac{r^m}{1-r^m}, \quad \text{thus}$$

$$\lim_{N\to\infty} |f_1(z) - f_{1,N}(z)| \leq \lim_{N\to\infty} \sum_{n=N}^{\infty} \frac{r^n}{1-r^n} = 0 \qquad (r<1).$$

Thus $f_{1,N} \longrightarrow f_1$ uniformly in compacts of $B_1(0)$ and f_1 is holomorphic. Since $f_{1,N}^{(k)} \xrightarrow{N\to\infty} f_1^{(k)}$ we observe that

$$f_1^{(k)} = \sum_{n=1}^{\infty} \left(\frac{z^n}{1-z^n}\right)^{(k)}.$$

$$f_2(z) = \int_0^1 e^{tz}(1-tz)^4 \, dt \quad \text{on } \mathbb{C}.$$

Observe that f_2 is continuous since $e^{tz}(1-tz)^4$ is a continuous function for $t \in [0,1]$ and $z \in \mathbb{C}$ and we are integrating in a compact set. Let Δ be any triangle in $\mathbb{C}$.

By Fubini's Thm

$$\int_\Delta \int_0^1 e^{tz}(1-tz)^4 \, dt = \int_0^1 \int_\Delta e^{tz}(1-tz)^4 dz = 0$$

Cauchy's Thm because $e^{tz}\cdot(1-tz)^4$ is holomorphic in $\mathbb{C}$.

By Morera's Thm f_2 is holomorphic.

Cauchy's formula yields:

$$f_2^{(m)}(z) = \frac{m!}{2\pi i}\int_C \frac{f_2(\zeta)}{(\zeta - z)^{m+1}}\, d\zeta \quad , \ C \text{ circle centered in } z.$$

$$= \frac{m!}{2\pi i}\int_C \frac{1}{(\zeta - z)^{m+1}} \int_0^1 (1-t\zeta)^4 e^{t\zeta}\, dt\, d\zeta$$

Fubini

$$= \int_0^1 \left\{ \frac{m!}{2\pi i}\int_C \frac{(1-t\zeta)^4 e^{t\zeta}}{(\zeta - z)}\, d\zeta \right\} dt$$

Cauchy's Thm

$$= \int_0^1 \left((1-tz)^4 e^{tz}\right)^{(m)} dt$$

$$f_3(z) = \sum_{m=0}^{\infty} m^2 \exp(2\pi i m^3 z) \qquad \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$$

We proceed like in f_1. Define $f_{3,N}(z) = \sum_{m=0}^{N} m^2 \exp(2\pi i m^3 z)$.

$\forall \ N \geq 1$, $f_{3,N}$ is holomorphic. Now let $\Omega_\varepsilon = \{z \in \mathbb{C} : \operatorname{Im}(z) \geq \varepsilon\}$, $\varepsilon > 0$. For any $z \in \Omega_\varepsilon$ we have

$$|f_3(z) - f_{3,N}(z)| \leq \sum_{m=N+1}^{\infty} m^2 e^{-2\pi \operatorname{Im}(z) m^3} \leq \sum_{m=N+1}^{\infty} m^2 e^{-2\pi \varepsilon m^3} \xrightarrow{N \to \infty} 0$$

since $\sum_{m=1}^{\infty} m^2 \cdot e^{-2\pi \varepsilon m^3} < +\infty$. Notice that the limit is uniform in Ω_ε.

We can write

$$f^{(k)}(z) = \sum_{n=1}^{\infty} n^{?}\left(e^{2\pi i n^3 z}\right)^{(k)}.$$

3. a) Let $z \in D_1(0)$, then $\inf_{\theta \in [0,2\pi]} |z - e^{i\theta}| = r > 0$. Thus

$$\left| \int_{\gamma} \frac{\tilde{g}(w)}{w-z}\, dw \right| \leq \frac{2\pi}{r} \cdot \sup_{\theta} |g(\theta)| < +\infty$$ so the

integral is well-defined.

b) Observe that f is continuous. Let $z \in D_1(0)$

and $r = \inf_{\theta \in [0,2\pi]} |z - e^{i\theta}|$. Let $\varepsilon > 0$, then $\forall$

$\zeta \in B_\varepsilon(z)$ we have

$$|f(z) - f(\zeta)| \leq \int_0^{2\pi} |g(\theta)| \frac{|z-\zeta|}{|e^{i\theta} - z| \cdot |e^{i\theta} - \zeta|}\, d\theta$$

$$\leq \sup_{\theta \in [0,2\pi]} |g(\theta)| \cdot 2\pi \cdot \frac{\varepsilon}{r(r-\varepsilon)}.$$

Now let Δ be a triangle in $D_1(0)$. We compute

$$\int_\Delta \int_\gamma \frac{1}{2\pi i} \frac{\tilde{g}(w)}{w-z}\, dw\, dz = \int_\gamma \frac{\tilde{g}(w)}{2\pi i} \int_\Delta \frac{1}{w-z}\, dz\, dw = 0$$

Fubini

$\frac{1}{w-z}$ holomorphic

+ Cauchy's Thm.

d) We use Cauchy's formula: let $z \in \Delta_1(0)$ and

$\tilde{\gamma} : [0,1] \longrightarrow \Delta_1(0)$

$t \longmapsto z + \varepsilon \cdot e^{2\pi i t}$ for $|z + \varepsilon \cdot e^{2\pi i t}| < 1 \quad \forall \ t \in [0,1]$.

Thus

$$f^{(1)}(z) = \frac{1}{2\pi i} \int_{\tilde{\gamma}} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta = \frac{1}{(2\pi i)^2} \int_{\tilde{\gamma}} \frac{1}{(\zeta - z)^2} \int_\gamma \frac{\tilde{g}(w)}{w - \zeta}\, dw\, d\zeta$$

Fubini

$$= \frac{1}{(2\pi i)^2} \int_\gamma \tilde{g}(w) \left\{ \int_{\tilde{\gamma}} \frac{1}{(\zeta - z)^2 \cdot (w - \zeta)}\, d\zeta \right\} dw$$

$$= \frac{1}{2\pi i} \int_\gamma \tilde{g}(w) \left\{ \frac{1}{2\pi i} \int_{\tilde{\gamma}} \frac{1/w-\zeta}{(\zeta - z)^2}\, d\zeta \right\} dw$$

Cauchy's formula

$$= \frac{1}{2\pi i} \int_\gamma \frac{\tilde{g}(w)}{(w - \zeta)^2}\, dw$$

4. a) $f(z) = \sin(z^2)$

Zeros: $\pm\sqrt{\pi k}$, $\pm i\sqrt{\pi k}$, $k \geq 1$ w/ order 1

0 w/ order 2

c) Let $a_{n+1} := 0$. The m-th coefficient of the Taylor expansion of $e^z - p(z)$ around 0 vanishes exactly when $a_m = 1/m!$.
So the order of the zero $z_0 = 0$ for $e^z - p(z)$ equals to

$$\min \{ 1 < m \leq n+1 \mid a_m \neq 1/m! \}.$$