Exercise sheet 6

Exercise worth bonus points: Exercise 5

1. (a) Let $\alpha \in \mathbf{C}$ be a fixed non-zero complex number. Construct a non-constant function $f \in \mathcal{H}(\mathbf{C})$ such that $f(z+\alpha) = f(z)$ for all $z \in \mathbf{C}$.
   Hint: consider first the case $\alpha = 2i\pi$.

   (b) Show that if $f \in \mathcal{H}(\mathbf{C})$ satisfies the relations
   $$f(z+1) = f(z)$$
   $$f(z+i) = f(z)$$
   for all $z \in \mathbf{C}$, then f is constant.

2. Let $U \subset \mathbf{C}$ be an open set and $z_0 \in U$. Let f be holomorphic on U outside z_0 with a pole of order $k \geqslant 1$ at z_0. Define
   $$g(z) = (z - z_0)^k f(z)$$
   for $z \neq z_0$.

   (a) Show that z_0 is a removable singularity of the function g.

   (b) Show that
   $$\operatorname{res}_{z_0}(f) = \lim_{\substack{z \to z_0 \\ z \neq z_0}} \frac{1}{(k-1)!} g^{(k-1)}(z).$$

3. Show that the following line integrals exist, and compute their values, where the curves are always oriented counterclockwise:

   (a)
   $$\int_\gamma \frac{\cos(z)}{z^2(z^2-8)} dz, \quad \gamma \text{ the boundary of the square } [-1,1] \times [-1,1]$$

   (b)
   $$\int_\gamma \frac{e^z}{e^{2z}-1} dz, \quad \gamma \text{ the boundary of the triangle with vertices } -1-i, 4i, 3.$$

   Hint: the previous exercise can be used to compute residues.

4. Show that the following functions are holomorphic in $\mathbf{C}$ except for isolated singularities. Show that these singularities are poles and determine their orders and residues.

   (a) $f(z) = \frac{1}{\cos(z^2)}$

   (b) $f(z) = \frac{z}{e^z - 1}$.

5. Consider the holomorphic function Γ defined in Exercise 1 of Exercise sheet 5. We showed that it is holomorphic on

$$U = \mathbf{C} - \{0, -1, -2, \ldots\}.$$

   (a) Show that the function $f\colon [0,1] \to \mathbf{R}$ defined by $f(t) = -1$ if $t = 0$ and $f(t) = (e^{-t} - 1)/t$ if $0 < t \leqslant 1$ is continuous.

   (b) Show that the function

$$g(z) = \int_0^1 f(t) t^z dt$$

   is defined and holomorphic for $\mathrm{Re}(z) > 0$.

   (c) Show that for $\mathrm{Re}(z) > 1$, we have

$$\Gamma(z) = \frac{1}{z} + g(z) + \int_1^{+\infty} e^{-t} t^{z-1} dt.$$

   (d) Deduce that Γ has a pole at $z = 0$ with residue 1.

   (e) Let $k \geqslant 0$ be an integer. Show that Γ has a simple pole at $-k$ with residue

$$\mathrm{res}_{-k}(\Gamma) = \frac{(-1)^k}{k!}.$$

   Hint: argue by induction using the relation $\Gamma(z+1) = z\Gamma(z)$.