Exercise sheet 6

Exercise worth bonus points: Exercise 5

1. (a) Let $\alpha \in \mathbf{C}$ be a fixed non-zero complex number. Construct a non-constant function $f \in \mathcal{H}(\mathbf{C})$ such that $f(z+\alpha) = f(z)$ for all $z \in \mathbf{C}$.
   Hint: consider first the case $\alpha = 2i\pi$.

   (b) Show that if $f \in \mathcal{H}(\mathbf{C})$ satisfies the relations

$$f(z+1) = f(z)$$
$$f(z+i) = f(z)$$

   for all $z \in \mathbf{C}$, then f is constant.

Solution:

(a) For $\alpha = 2\pi i$ we can take $f(z) = e^z$. It is clear that

$$f(z) = e^z = e^{z+2\pi i} = f(z + 2\pi i).$$

For $\alpha = 0$ the result is trivial and for any $\alpha \in \mathbf{C} \setminus \{0\}$ we can take $f(z) = e^{\frac{2\pi i}{\alpha} z}$.

(b) We call $Q = [-1, 1]^2$ the closed square centered in 0 of side lenght 1. Since Q is compact it holds that

$$\sup_{z \in Q} |f(z)| =: B < \infty.$$

Now let $z \in \mathbf{C}$. There exists $n, k \in \mathbf{Z}$ such that $z + n + ik \in Q$. Using the relations that f satifies inductively we get that

$$f(z) = f(z + n + ik) \Rightarrow |f(z)| \leqslant B.$$

We conclude that f is bounded and since it is holomorphic in the whole complex plane, by Liouville's theorem f has to be constant.

2. Let $U \subset \mathbf{C}$ be an open set and $z_0 \in U$. Let f be holomorphic on U outside z_0 with a pole of order $k \geqslant 1$ at z_0. Define

$$g(z) = (z - z_0)^k f(z)$$

for $z \neq z_0$.

(a) Show that z_0 is a removable singularity of the function g.

(b) Show that

$$\operatorname{res}_{z_0}(f) = \lim_{\substack{z \to z_0 \\ z \neq z_0}} \frac{1}{(k-1)!} g^{(k-1)}(z),$$

where $g(z_0)$ is the value at z_0 of the unique holomorphic function on U that extends g.

Solution:

(a) If f is holomorphic on U and has a pole of order k in z_0 we know that for $z \in B_r(z_0) \setminus \{z_0\} \subset U$ we can write

$$f(z) = \frac{a_k}{(z-z_0)^k} + \cdots + \frac{a_1}{z-z_0} + g(z),$$

for g holomorphic in $B_r(z_0)$. Thus, for $z \in B_r(z_0) \setminus \{z_0\}$ it holds

$$g(z) = a_k + a_{k-1}(z-z_0) + \cdots + a_1(z-z_0)^{k-1} + (z-z_0)^k g(z).$$

Since $a_k + a_{k-1}(z-z_0) + \cdots + a_1(z-z_0)^{k-1} + (z-z_0)^k g(z)$ is holomorphic in $B_r(z_0)$ and coincides with f for $z \in B_r(z_0) \setminus \{z_0\}$ we conclude that z_0 is a removable singularity of g.

(b) We know that $\operatorname{res}_{z_0}(f) = a_1$. Observe that

$$\begin{aligned} g^{(k-1)}(z) &= a_1 \cdot (k-1)! + \sum_{n=0}^{k-1} \binom{k-1}{n} k \cdot (k-1) \ldots (k-n+1)(z-z_0)^{k-n} g^{(k-1-n)}(z) \\ &= a_1 \cdot (k-1)! + (z-z_0)h(z), \end{aligned}$$

where h is holormophic in U. Thus

$$a_1 = \lim_{\substack{z \to z_0 \\ z \neq z_0}} \frac{1}{(k-1)!} g^{(k-1)}(z),$$

as we wanted.

3. Show that the following line integrals exist, and compute their values, where the curves are always oriented counterclockwise:

(a)

$$\int_\gamma \frac{\cos(z)}{z^2(z^2-8)} dz, \quad \gamma \text{ the boundary of the square } [-1,1] \times [-1,1]$$

(b)

$$\int_\gamma \frac{e^z}{e^{2z}-1}dz, \quad \gamma \text{ the boundary of the triangle with vertices } -1-i, 4i, 3.$$

Hint: the previous exercise can be used to compute residues.

Solution:

(a) Observe that the integral is well-defined since the only singularities of the function we are integrating are $\sqrt{8}, -\sqrt{8}$ and 0, which are not γ. Since $z = 0$ is the only singularity that lies inside the region bordered by γ it is enough to check its behaviour. Observe that

$$f(z) =: \frac{\cos(z)}{z^2(z^2-8)} = \frac{1}{z^2}\left(\frac{\cos(z)}{z^2-8},\right)$$

and $\cos(0)/(0^2 - 8) = -\frac{1}{8} \neq 0$, so we can take a small neighborhood of 0 where $\cos(z)/(z^2 - 8) \neq 0$ and conclude that $z = 0$ is a pole of order 2 of f. We use the formula from the previous exercise to compute the residue:

$$\lim_{\substack{z\to 0\\ z\neq 0}} -\frac{(z^2-8)\sin(z) + 2z\cos(z)}{(z^2-8)^2} = 0,$$

thus

$$\int_\gamma \frac{\cos(z)}{z^2(z^2-8)}dz = 0.$$

(b) Let $f(z) = \frac{e^z}{e^{2z}-1}$ and observe that f has singularities for $z = \pi i k$, $k \in \mathbf{Z}$, since $e^z \neq 0$, $\forall z \in \mathbf{C}$. The singularities that lie inside of the region given by γ are $z = 0$ and $z = \pi i$. Observe that

$$\frac{e^z}{e^{2z}-1} = \frac{1}{2z}\left(\frac{e^z}{1 + \frac{2z}{2!} + \sum_{n=2}^{\infty} \frac{(2z)^n}{(n+1)!}}\right). \tag{1}$$

Since e^z is never zero and we can take $\varepsilon > 0$ small enough so that $\forall z \in B_\varepsilon(0)$

$$\left|\frac{2z}{2!} + \sum_{n=2}^{\infty} \frac{(2z)^n}{n!}\right| < \frac{1}{2}$$

we can conclude from equation 1 that $z = 0$ is a pole of order 1 of f. A similar analysis shows that πi is also a pole of order 1 of f. We compute the residue at 0 using equation 1 and the analogous equation to compute the residue at πi:

$$\lim_{z\to 0} \frac{ze^z}{e^{2z}-1} = \frac{1}{2}$$
$$\lim_{z\to \pi i} \frac{(z-\pi i)e^z}{e^{2z}-1} = -\frac{1}{2},$$

so by the Residue's Theorem we have

$$\int_\gamma f(z)dz = 2\pi i(\mathrm{res}_0(f) + \mathrm{res}_{\pi i}(f)) = 0.$$

4. Show that the following functions are holomorphic in $\mathbf{C}$ except for isolated singularities. Show that these singularities are poles and determine their orders and residues.

   (a) $f(z) = \frac{1}{\cos(z^2)}$

   (b) $f(z) = \frac{z}{e^z - 1}$.

   Solution:

   (a) We note that $\cos(z^2) = 0 \Leftrightarrow z^2 = \frac{\pi}{2} + \pi k$ for $k \in \mathbf{Z}$. Taking the square roots and defining

   $$z_k = \sqrt{\frac{\pi}{2} + \pi k}, \text{ for } k \geqslant 0,$$

   we observe that the singularies are given by $\{z_k, -z_k, iz_k, -iz_k\}_{k=0}^\infty$. Let $w_k \in \{z_k, -z_k, iz_k, -iz_k\}$ and observe that

   $$\frac{1}{\cos(z^2)} = \frac{1}{-2w_k \sin(w_k^2)(z - w_k)} \frac{1}{1 + \sum_{n=2}^\infty a_n^{w_k}(z - w_k)^{n-1}},$$

   and if we take $\varepsilon > 0$ small enough so that $\forall z \in B_\varepsilon(w_k)$ it holds that

   $$\left| \sum_{n=2}^\infty a_n^{w_k}(z - w_k)^{n-1} \right| < \frac{1}{2}$$

   we can conclude that w_k is a pole of order one of f. We compute the residue:

   $$\lim_{z\to w_k} \frac{z - w_k}{\cos(z^2)} = -\frac{1}{2w_k \sin(w_k^2)}.$$

   Observe that $\sin(z_k^2) = \sin((-z_k)^2) = 1$ and $\sin((iz_k)^2) = \sin((-iz_k)^2) = -1$.

(b) As we've seen in Exercise Sheet 4, the singularity at $z = 0$ is removable. The other singularies of f are given by $z_k = 2\pi ik$ for $k \in \mathbf{Z} \setminus \{0\}$. Expanding in Taylor series we get

$$f(z) = \frac{1}{z - 2\pi ik} \cdot \frac{z}{1 + \sum_{n=2}^{\infty} \frac{(z-2\pi ik)^{n-1}}{n!}}, \tag{2}$$

for $\varepsilon > 0$ small enough we know that $\forall z \in B_\varepsilon(2\pi ik)$

$$\left| \sum_{n=2}^{\infty} \frac{(z - 2\pi ik)^{n-1}}{n!} \right| < \frac{1}{2},$$

so we conclude that the singularities are poles of order 1 and using equation 2 we get that

$$\lim_{z \to 2\pi ik} \frac{z(z - 2\pi ik)}{e^z - 1} = 2\pi ik$$

thus $\operatorname{res}_{z_k}(f) = z_k$.

5. Consider the holomorphic function Γ defined in Exercise 1 of Exercise sheet 5. We showed that it is holomorphic on

$$U = \mathbf{C} - \{0, -1, -2, \ldots\}.$$

(a) Show that the function $f\colon [0,1] \to \mathbf{R}$ defined by $f(t) = -1$ if $t = 0$ and $f(t) = (e^{-t} - 1)/t$ if $0 < t \leqslant 1$ is continuous.

(b) Show that the function

$$g(z) = \int_0^1 f(t) t^z dt$$

is defined and holomorphic for $\mathrm{Re}(z) > 0$.

(c) Show that for $\mathrm{Re}(z) > 1$, we have

$$\Gamma(z) = \frac{1}{z} + g(z) + \int_1^{+\infty} e^{-t} t^{z-1} dt.$$

(d) Deduce that Γ has a pole at $z = 0$ with residue 1.

(e) Let $k \geqslant 0$ be an integer. Show that Γ has a simple pole at $-k$ with residue

$$\operatorname{res}_{-k}(\Gamma) = \frac{(-1)^k}{k!}.$$

Hint: argue by induction using the relation $\Gamma(z + 1) = z\Gamma(z)$.

Solution:

(a) The function is clearly continuous for $0 < t \leqslant 1$, so we just have to show that it is also continuous at zero.

$$\lim_{t\to 0^+} \frac{e^{-t}-1}{t} = \lim_{t\to 0^+} \frac{-t+O(t^2)}{t} = -1$$

(b) Let $\Omega = \{z : \mathrm{Re}(z) > 1\}$. Observe that $F : [0,1] \times \Omega$, $F(t,z) = \frac{e^{-1}-1}{t} \cdot t^z$ is a continuous function and for every fixed $t_0 \in [0,1]$, $F(t_0, z)$ defines an holomoprhic function in Ω. Thus, by Theorem 5.4 we can conclude that g is holormophic in Ω.

(c) Recall that for $\mathrm{Re}(z) > 1$:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

and that the integral is absolutely convergent in this domain. We can split it as follows:

$$\Gamma(z) = \int_0^1 e^{-t} t^{z-1} dt + \int_1^\infty e^{-t} t^{z-1} dt.$$

To conclude, observe that if $-1 < \sigma < \infty$ then

$$\int_0^1 t^\sigma dt < \infty$$

and we also know that

$$\left| \int_0^1 e^{-t} t^{z-1} dt \right| < \infty$$

for $\mathrm{Re}(z) > 0$ so g can be written as

$$\int_0^1 \frac{e^{-t}-1}{t} t^z dt = \int_0^1 e^{-t} t^{z-1} dt - \int_0^1 t^{z-1} dt = \int_0^1 e^{-t} t^{z-1} dt - \frac{1}{z},$$

given the identity that we desired.

(d) Recall that $z\Gamma(z) = \Gamma(z+1)$, so for $\mathrm{Re}(z) > -1$ we can write

$$\Gamma(z) = \frac{\Gamma(z+2)}{z(z+1)}$$

and from the formula of the previous item we can compute $\Gamma(2) = 1$ so we conclude that $z = 0$ is a pole of order 1 since $\Gamma(z+2)/(z+1)$ is never zero in a small neighborhood of 0. Using the same identity we compute $\lim_{z\to 0} z\Gamma(z)$:

$$\lim_{z\to 0} z\Gamma(z) = \lim_{z\to 0} \frac{\Gamma(z+2)}{(z+1)} = \Gamma(2) = 1,$$

as we wanted.

Computation of $\Gamma(2)$:

$$\begin{aligned} \Gamma(2) &= \frac{1}{2} + \int_0^1 \frac{e^{-t}-1}{t} dt + \int_1^\infty e^{-t} t dt \\ &= \frac{1}{2} - \int_0^1 t dt + \int_0^\infty e^{-t} t dt \\ \frac{1}{2} - \frac{1}{2} + 1 = 1. \end{aligned}$$

(e) Using the identity $\Gamma(z+1) = z\Gamma(z)$ inductively we get, for $k \geqslant 0$.

$$\Gamma(z) = \frac{\Gamma(z+k+2)}{(z+k+1)(z+k)\dots(z+1)z},$$

and the identity holds for $\{z : -k-1 < \mathrm{Re}(z)\} \setminus \{-k\}$.

Using the same strategy as in the previous item we conclude that $z = -k$ is a pole of order 1 of Γ.

We compute the residue:

$$\begin{aligned} \lim_{z\to -k} \Gamma(z)(z+k) &= \lim_{z\to -k} \frac{\Gamma(z+k+2)}{(z+k+1)\dots(z+1)z} \\ &= \frac{\Gamma(2)}{1\cdot(-1)\dots(-k+1)(-k)} = \frac{(-1)^k}{k!}. \end{aligned}$$