Exercise sheet 7

Exercise worth bonus points: Exercise 3

1. Show that for $a > 0$, we have
$$\int_{-\infty}^{+\infty} \frac{\cos(x)}{x^2 + a^2} dx = \frac{\pi e^{-a}}{a}.$$
Hint: Observe that
$$\int_{-R}^{R} \frac{\sin(x)}{x^2 + a^2} dx = 0$$
for all $R > 0$.

2. Let $k \geqslant 1$ be an integer and $x > 0$ a real number. Compute
$$\mathrm{res}_{z=0} \left(\frac{x^z}{z^k} \right)$$
as a function of x, where $x^z = \exp(z \log(x))$ for all $z \in \mathbf{C}$.

3. Let f be a meromorphic function on $\mathbf{C}$. Define $g(z) = f(1/z)$ for $z \neq 0$ in $\mathbf{C}$.
   (a) Show that $g \in \mathcal{M}(\mathbf{C}^*)$.

   We assume from now on that g has a pole at $z_0 = 0$.

   (b) Show that f has only finitely many poles in $\mathbf{C}$.
   (c) Show that there exist polynomials p_1 and q_1, with $q_1 \neq 0$, and a real number $R > 0$, such that the meromorphic function $f - p_1/q_1$ is holomorphic and bounded for $|z| > R$.
   Hint: consider the principal part of g.
   (d) Show that there exist polynomials p_2 and q_2, with $q_2 \neq 0$ such that the meromorphic function $f - p_1/q_1 - p_2/q_2$ is holomorphic and bounded on $\mathbf{C}$.
   (e) Conclude that there exist polynomials p_3 and q_3, with $q_3 \neq 0$ such that $f = p_3/q_3$.

4. Let $f \in \mathcal{H}(\mathbf{C})$ be a non-constant holomorphic function. Show that for any $w \in \mathbf{C}$ and any $\delta > 0$, there exists $z \in \mathbf{C}$ such that $|f(z) - w| < \delta$.
Hint: if this were not true, consider the function $g(z) = 1/(f(z) - w)$.

5. Let $f \in \mathcal{H}(D_1(0))$. We assume that $f(0) = 0$ and that $|f(z)| \leqslant 1$ for all $z \in D_1(0)$.

   (a) Show that the function $g\colon D_1^*(0) \to \mathbf{C}$ defined by $g(z) = f(z)/z$ is holomorphic on $D_1^*(0)$ with a removable singularity at 0. We denote still by g the holomorphic extension of g to $D_1(0)$.

   (b) Let $r \in]0, 1[$. Show that $|g(z)| \leqslant 1/r$ if $|z| < r$.

   (c) Deduce that $|f(z)| \leqslant |z|$ for all $z \in D_1(0)$.