Exercise sheet 7

Exercise worth bonus points: Exercise 3

1. Show that for $a > 0$, we have
$$\int_{-\infty}^{+\infty} \frac{\cos(x)}{x^2 + a^2} dx = \frac{\pi e^{-a}}{a}.$$
Hint: Observe that
$$\int_{-R}^{R} \frac{\sin(x)}{x^2 + a^2} dx = 0$$
for all $R > 0$.

Solution:

Denote $g(z) = \frac{e^{iz}}{z^2+a^2}$. For $R > a + 1$ we integrate g in the closed path γ, given by the boundary of the upper half of the disk of radius R with center zero, in the counter-clockwise direction. The Residue Theorem gives us
$$\int_\gamma \frac{e^{iz}}{z^2 + a^2} dz = 2\pi i \operatorname{Res}(g, ia).$$
On the otherside, using line integrals we get
$$\int_\gamma \frac{e^{iz}}{z^2 + a^2} dz = \int_{-R}^{R} \frac{e^{ix}}{x^2 + a^2} dx + \int_0^\pi \frac{e^{iR(\cos(t)+i\sin(t))} Re^{it}}{R^2 e^{2it} + a^2} dt.$$
Observe that
$$\left| \int_0^\pi \frac{e^{iR(\cos(t)+i\sin(t))} Re^{it}}{R^2 e^{2it} + a^2} dt \right| \leqslant \int_0^\pi \frac{Re^{-R\sin(t)}}{R^2 - a^2} dt \leqslant \frac{\pi R}{R^2 - a^2},$$
where we used $\sin(t) > 0$ for $t \in [0, \pi]$ and $|R^2 e^{2it} + a^2| \geqslant R^2 - a^2$. Letting $R \to \infty$ and using that $\int_{-R}^{R} \frac{\sin(x)}{x^2+a^2} dx = 0$, we conclude that
$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + a^2} dx = \operatorname{Re}(2\pi i \operatorname{Res}(g, ia)).$$
To compute the residue we write
$$\frac{1}{z^2 + a^2} = \frac{1}{2ai} \cdot \frac{1}{x - ai} - \frac{1}{2ai} \cdot \frac{1}{x + ai},$$
so
$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + a^2} dx = \frac{\pi e^{-a}}{a}.$$

2. Let $k \geqslant 1$ be an integer and $x > 0$ a real number. Compute

$$\mathrm{res}_{z=0}\Big(\frac{x^z}{z^k}\Big)$$

as a function of x, where $x^z = \exp(z\log(x))$ for all $z \in \mathbf{C}$.

Solution:

Observe that $z = 0$ is a pole of order k. We use the formula computed in the previous exercise sheet to compute the residue:

$$\mathrm{res}_{z=0}\Big(\frac{x^z}{z^k}\Big) = \frac{1}{(k-1)!}\lim_{\substack{z\to 0\\ z\neq 0}}(e^{z\log x})^{(k-1)} = \frac{(\log x)^{k-1}}{(k-1)!}$$

3. Let f be a meromorphic function on $\mathbf{C}$. Define $g(z) = f(1/z)$ for $z \neq 0$ in $\mathbf{C}$.

   (a) Show that $g \in \mathcal{M}(\mathbf{C}^*)$.

   We assume from now on that g has a pole at $z_0 = 0$.

   (b) Show that f has only finitely many poles in $\mathbf{C}$.

   (c) Show that there exist polynomials p_1 and q_1, with $q_1 \neq 0$, and a real number $R > 0$, such that the meromorphic function $f - p_1/q_1$ is holomorphic and bounded for $|z| > R$.

   Hint: consider the principal part of g.

   (d) Show that there exist polynomials p_2 and q_2, with $q_2 \neq 0$ such that the meromorphic function $f - p_1/q_1 - p_2/q_2$ is holomorphic and bounded on $\mathbf{C}$.

   (e) Conclude that there exist polynomials p_3 and q_3, with $q_3 \neq 0$ such that $f = p_3/q_3$.

Solutions:

(a) Let $V = \{z \in \mathbf{C} \setminus \{0\} : g(z) = \infty\}$ and $U = \{z \in \mathbf{C} \setminus \{0\} : f(z) = \infty\}$ and consider K a compact set in $\mathbf{C} \setminus \{0\}$. Observe that there exists $\varepsilon > 0$ and $K > 0$ such that $\forall z \in K$, $\varepsilon \leqslant |z| \leqslant K$. Thus

$$V \cap K \subset \{z : \frac{1}{K} \leqslant |z| \leqslant \frac{1}{\varepsilon}\} \cap U,$$

which has to be finite because f is meremorphic.

Since $1/z$ is holomorphic in $\mathbf{C} \setminus \{0\}$ and f is holomorphic in U it follows that $f(1/z)$ has to be holormorphic in V. And, if z_0 is such that $g(z_0) = \infty$ then

f has a pole in $1/z_0$, thus z_0 has to be a pole of g: for $\varepsilon > 0$ sufficiently small such that $|z_0| > \varepsilon$ and

$$f(z) = \frac{h(z)}{(z - \frac{1}{z_0})^k}$$

for all $z \in B_{\varepsilon/2}(\frac{1}{z_0})$, thus

$$g(z) = \frac{z^k z_0^k h(1/z)}{(z_0 - z)^k}$$

(b) If z_0 is a pole of g, there exists $\varepsilon > 0$ such that $\forall z \in B_\varepsilon(0) \setminus \{0\}$ it holds that

$$f(1/z) = g(z) = \frac{h(z)}{z^k}$$

for h holormorphic and non-zero in $B_\varepsilon(0)$. Thus, for $|1/z| \leqslant \varepsilon$, that is $|z| \geqslant \varepsilon$ f is holormophic. Take $K = B_{\frac{2}{\varepsilon}}(0)$. Then $\{z : f(z) = \infty\} = \{z : f(z) = \infty\} \cap K$ which is finite, as we wanted.

(c) Since $z_0 = 0$ is a pole of order $k \geqslant 0$ of g we can write, for a $\delta > 0$

$$f(1/z) = g(z) = \frac{a_k}{z^k} + \cdots \frac{a_1}{z} + h(z),$$

for all $z \in B_\delta(0) \setminus \{0\}$ where h is holormophic in $B_\delta(0)$. This implies that h is bounded for $|z| \leqslant \delta/2$. So we take $R = \frac{2}{\delta}$ and observe that whenever $|z| > R$:

$$f(z) - a_k z^k + \cdots a_1 z$$

is holormophic and bounded for $|z| > R$.

(d) Now let z_0 be a pole of order l of f, we know that there exists $\varepsilon > 0$, $|z_0| + \varepsilon < R$, t holomorphic in $B_\varepsilon(z_0)$ such that

$$f(z) - \frac{a_l}{(z - z_0)^l} - \cdots - \frac{a_1}{z - z_0} = t(z)$$

holds in $\forall z \in B_\varepsilon(z_0) \setminus \{z_0\}$. Observe that t is bounded in $B_{\varepsilon/2}(z_0)$ and that for $|z - z_0| \geqslant \varepsilon/2$ it holds that

$$\left| \frac{a_l}{(z - z_0)^l} + \cdots + \frac{a_1}{z - z_0} \right| \leqslant \sum_{n=1}^{l} |a_n| \frac{2}{\varepsilon}.$$

We denote by $h_{z_0}(z) = \frac{a_l}{(z-z_0)^l} + \cdots + \frac{a_1}{z-z_0}$ and let

$$\frac{p_2(z)}{q_2(z)} = \sum_{z_0 \text{ pole of } f} h_{z_0}(z).$$

Now we observe that $f - p_1(z) - \frac{p_2(z)}{q_2(z)}$ is bounded. Since $f - p_1(z)$ and $\frac{p_2(z)}{q_2(z)}$ are bounded for $|z| > K$ we can use triangle inequality in this region. In small balls around the poles we know that $f - \frac{p_2(z)}{q_2(z)}$ is bounded and $p_1(z)$ also is. For z in the complement of the all the regions considered we know that $f - p_1(z) - \frac{p_2(z)}{q_2(z)}$ is holormorphic and since the region is closed and bounded the function is bounded as well.

(e) Since $f - p_1(z) - \frac{p_2(z)}{q_2(z)}$ is holormophic and bounded it must be constant by Liouville's theorem. Thus

$$f = c\left(p_1(z) - \frac{p_2(z)}{q_2(z)}\right) = \frac{p_3(z)}{q_3(z)}.$$

4. Let $f \in \mathcal{H}(\mathbf{C})$ be a non-constant holomorphic function. Show that for any $w \in \mathbf{C}$ and any $\delta > 0$, there exists $z \in \mathbf{C}$ such that $|f(z) - w| < \delta$.

Hint: if this were not true, consider the function $g(z) = 1/(f(z) - w)$.

Solution:

(Observe that we are asked to prove that $f(\mathbf{C})$ is dense in $\mathbf{C}$.)

By contradiction, suppose there exists $w \in \mathbb{C}$ and $\delta > 0$ with $B_\delta(w) \cap f(\mathbb{C}) = \varnothing$. Then $|f(z) - w| \geqslant \delta$ for all $z \in \mathbb{C}$ and thus the function is

$$g : \mathbb{C} \to \mathbb{C},\ z \mapsto \frac{1}{f(z) - w}$$

holomorphic with $|g(z)| \leqslant \delta^{-1}$ for all $z \in \mathbb{C}$, thus bounded; consequently, according to Liouville g is constant. But then f is also constant, which is a contradiction.

5. Let $f \in \mathcal{H}(D_1(0))$. We assume that $f(0) = 0$ and that $|f(z)| \leqslant 1$ for all $z \in D_1(0)$.

(a) Show that the function $g\colon D_1^*(0) \to \mathbf{C}$ defined by $g(z) = f(z)/z$ is holomorphic on $D_1^*(0)$ with a removable singularity at 0. We denote still by g the holomorphic extension of g to $D_1(0)$.

(b) Let $r \in]0, 1[$. Show that $|g(z)| \leqslant 1/r$ if $|z| < r$.

(c) Deduce that $|f(z)| \leqslant |z|$ for all $z \in D_1(0)$.

Solution:

(a) Let $g(z) = f(z)/z$ in $\Omega \setminus \{0\}$ and observe that, since f is differentiable in 0, we have

$$\lim_{z \to 0} \frac{f(z)}{z} = f'(0),$$

thus the singularity must be removable.

(b) Consider $B_r(0)$, with $0 < r < 1$. By the maximum principle we know that g attains its maximum in the border, thus

$$\max_{z \in B_r(0)} |g(z)| = |g(z_r)| = \frac{|f(z_r)|}{|z_r|} \leqslant \frac{1}{r}.$$

(c) Letting $r \to 1$ in the inequality above we get that $|g(z)| \leqslant 1$ so $|f(z)| \leqslant |z|$ $\forall z \in D_1(0)$.