Exercise sheet 8

Exercise worth bonus points: Exercise 1

1. For $z \in \mathbf{C}$ such that $\sin(z) \neq 0$, we denote

$$\operatorname{cotan}(z) = \frac{\cos(z)}{\sin(z)} = i\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}.$$

   (a) Show that $\operatorname{cotan} \in \mathcal{M}(\mathbf{C})$ and determine its poles and the corresponding residues.

   (b) Let u be a complex number which is not an integer, and let

$$f(z) = \frac{\pi \operatorname{cotan}(\pi z)}{(u+z)^2}.$$

   Show that $f \in \mathcal{M}(\mathbf{C})$ and determine its poles and the corresponding residues.

   (c) Let $n \geqslant 1$ be an integer such that $n > |u|$. Compute the integral

$$\int_{\gamma_n} f(z)dz$$

   where γ_n is the circle of radius $n + 1/2$ oriented counterclockwise.

   (d) Deduce that

$$\lim_{n\to+\infty} \sum_{k=-n}^{n} \frac{1}{(u+k)^2} = \frac{\pi^2}{(\sin \pi u)^2}.$$

2. Let w_0 be a complex number such that $|w_0| < 1$. Show that

$$B(z) = \frac{w_0 - z}{1 - \bar{w}_0 z}$$

   defines a function with the following properties:

   (a) It is a holomorphic function on $D_1(0)$ with values in $D_1(0)$;

   (b) $B(w_0) = 0$ and $B(0) = w_0$;

   (c) $|B(z)| = 1$ if $|z| = 1$;

   (d) B is bijective from $D_1(0)$ to $D_1(0)$.

3. Let $U \subset \mathbf{C}$ be an open set containing the closed unit disc $\bar{D}_1(0)$. Let $f \in \mathcal{H}(U)$, and assume that f is not constant. Suppose further that $|f(z)| = 1$ if $|z| = 1$.

   (a) Show that $m = \min_{|z| \leqslant 1} |f(z)|$ exists, and that it is strictly less than 1. (Hint: show that $m \leqslant 1$, and that f would be constant if there was equality, using the maximum modulus principle.)

   (b) Show that $m = 0$. (Hint: if $m > 0$, prove that we would have $m = 1$, by considering the function $g = 1/f$.)

   (c) Deduce that there exists $z \in D_1(0)$ such that $f(z) = 0$.

   (d) Show that for any $w \in D_1(0)$, there exists z such that $f(z) = w$. (Hint: apply the previous question to an auxiliary function constructed using Exercise 2.)

4. Let $r > 1$ be a real number and let f and g be functions holomorphic in $D_r(0)$. We assume that for $|z| \leqslant 1$, we have $f(z) = 0$ if and only if $z = 0$, and that $\mathrm{ord}_0(f) = 1$. We also assume that g is not the zero function.

   For $\varepsilon \in \mathbf{C}$, we denote
   $$f_\varepsilon(z) = f(z) + \varepsilon g(z).$$

   (a) Show that there exists a real number $\delta > 0$ such that we have $|f(z)/g(z)| \geqslant \delta$ if z satisfies $|z| = 1$ and $g(z) \neq 0$.

   (b) Show that if $|\varepsilon| < \delta$, the function f_ε is holomorphic on $D_r(0)$ and there is a unique z_ε such that $|z_\varepsilon| \leqslant 1$ and $f_\varepsilon(z_\varepsilon) = 0$. We denote by Z the map from $D_\delta(0)$ to $\bar{D}_1(0)$ such that $Z(\varepsilon) = z_\varepsilon$.

   (c) Prove that for $|\varepsilon| < \delta$, we have $|z_\varepsilon| < 1$.

   (d) Let (ε_n) be a sequence with $|\varepsilon_n| < \delta$ which converges to $\varepsilon \in D_\delta(0)$. If $Z(\varepsilon_n)$ converges to some complex number z, show that $z = Z(\varepsilon)$. (Hint: use the uniqueness of $Z(\varepsilon)$.)

   (e) Show that Z is continuous. (Hint: use the following fact from analysis: if a bounded sequence (w_n) of complex numbers has the property that all convergent subsequences (w_{n_k}) have *the same limit* w, then (w_n) converges to w.)