Exercise sheet 8

Exercise worth bonus points: Exercise 1

1. For $z \in \mathbf{C}$ such that $\sin(z) \neq 0$, we denote
$$\operatorname{cotan}(z) = \frac{\cos(z)}{\sin(z)} = i\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}.$$

(a) Show that cotan $\in \mathcal{M}(\mathbf{C})$ and determine its poles and the corresponding residues.

(b) Let u be a complex number which is not an integer, and let
$$f(z) = \frac{\pi \operatorname{cotan}(\pi z)}{(u+z)^2}.$$
Show that $f \in \mathcal{M}(\mathbf{C})$ and determine its poles and the corresponding residues.

(c) Let n be an integer such that $|n| > |u|$. Compute the integral
$$\int_\gamma f(z)dz$$
where γ_n is the circle of radius $n + 1/2$ oriented counterclockwise.

(d) Deduce that
$$\lim_{n \to +\infty} \sum_{k=-n}^{n} \frac{1}{(u+k)^2} = \frac{\pi^2}{(\sin \pi u)^2}.$$

Solution:

(a) Observe that $\sin(z) = 0 \Leftrightarrow z = k\pi$, for $k \in \mathbf{Z}$. Denote $U = \{z \in \mathbf{C} : z \neq k\pi\}$ and observe that $\operatorname{cotan}(z)$ is holomorphic in U. For $z = \pi k$ we have $\lim_{z \to \pi k} \frac{\cos(z)}{\sin(z)} = +\infty$. Also, given any compact $K \subset \mathbf{C}$ we have $|K \cup (\mathbf{C} \setminus U)| < \infty$, so we conclude that cotan is a meromorphic function in Cc.
The poles of the function are given by $z = \pi k$ for $k \in \mathbf{Z}$ and we compute the residues:

$$\frac{\cos(z)}{\sin(z)} = \frac{(-1)^k + (-1)^{k+1}\frac{(z-\pi k)^2}{2!} + O((z-\pi k)^4)}{(-1)^k(z-\pi k) + (-1)^{k+1}\frac{(z-\pi k)^3}{3!} + O((z-\pi k)^5)},$$

thus,

$$\lim_{z\to\pi k}(z-\pi k)\frac{\cos(z)}{\sin(z)}=1,$$

and we conclude that $\mathrm{res}_{\pi k}(f)=1$.

(b) Observe that $\pi\,\mathrm{cotan}(\pi z)$ and $\frac{1}{(u+z)^2}$ are meromorphic functions in $\mathbf{C}$ so we conclude that f is also meromorphic in $\mathbf{C}$. The poles of the function are given by $\{z=k, k\in\mathbf{Z}\}\cup\{-u\}$ and

$$\mathrm{res}_k(f)=\lim_{\substack{z\to k\\ z\neq k}}(z-\pi k)\frac{\pi\,\mathrm{cotan}(\pi z)}{(u+z)^2}=\frac{1}{(u+k)^2}.$$

Observe that $-u$ is a pole of order 2 if $u\neq n+\frac{1}{2}$ with residue given by:

$$\lim_{\substack{z\to -u\\ z\neq -u}}(\pi\,\mathrm{cotan}(\pi z))'=-\frac{\pi^2}{\sin^2(\pi u)}.$$

If $u=n+\frac{1}{2}$ then the pole is of order 1 and the residue is given by:

$$\lim_{\substack{z\to -u\\ z\neq -u}}\frac{\pi\,\mathrm{cotan}(\pi z)}{u+z}=-\pi^2=-\frac{\pi^2}{\sin^2(\pi u)}.$$

(c) Observe that γ_n doesn't intersect with any of f poles. So, by the Residue's Theorem we conclude that

$$\int_{\gamma_n}f(z)dz=2\pi i\left(\sum_{|k|<n}\frac{1}{(u+k)^2}-\frac{\pi^2}{\sin^2(\pi u)}\right)$$

(d) To prove the result it is enough to show that

$$\lim_{n\to\infty}\int_{\gamma_n}f(z)dz=0.$$

First observe that $\cot(\pi z)$ is bounded for $\mathrm{Im}(z)>1$. It holds that

$$|\,\mathrm{cotan}(\pi z)|=\left|\frac{i(e^{2iz}+1)}{e^{2iz}-1}\right|\leqslant\frac{e^{-2\,\mathrm{Im}(z)}+1}{1-e^{-2\,\mathrm{Im}(z)}}\leqslant C_1$$

If $\mathrm{Im}(z) < -1$ then

$$|\operatorname{cotan}(\pi z)| = \left|\frac{i(e^{2iz}+1)}{e^{2iz}-1}\right| \leqslant \frac{e^{2\,\mathrm{Im}(z)}+1}{e^{2\,\mathrm{Im}(z)}-1} \leqslant C_2.$$

We also observe that if z is of the form $N+\frac{1}{2}+it$ or $-N-\frac{1}{2}+it$, with $t \in \mathbb{R}$ we can also control $\operatorname{cotan}(\pi z)$.

$$\left|\operatorname{cotan}\left(\pi\left(N+\frac{1}{2}+it\right)\right)\right| = \left|\frac{-e^{-2\pi t}+1}{-e^{-2\pi t}-1}\right| \leqslant C'.$$

We can bound $\operatorname{cotan}(\pi z)$ in a vertical strip centered $n+1/2$ uniformly in n, and we know that for n big enough $\gamma_n(t)$, for $|\mathrm{Im}(\gamma_n(t))| < 1$ is contained in this strip. Thus,

$$\left|\int_{\gamma_n} f(z)dz\right| \leqslant \tilde{C}\frac{n+\frac{1}{2}}{(n+\frac{1}{2}-|u|)^2},$$

which goes to zero as $n \to \infty$.

2. Let w_0 be a complex number such that $|w_0| < 1$. Show that

$$B(z) = \frac{w_0 - z}{1-\bar{w}_0 z}$$

defines a function with the following properties:

(a) It is a holomorphic function on $D_1(0)$ with values in $D_1(0)$;

(b) $B(w_0) = 0$ and $B(0) = w_0$;

(c) $|B(z)| = 1$ if $|z| = 1$;

(d) B is bijective.

Solution:

First observe that $1-\overline{w_0}z = 0 \Leftrightarrow z = \frac{1}{\overline{w_0}}$ and since $|w_0| < 1$ it holds that $\frac{1}{|w_0|} > 1$, so B is holormophic in $D_1(0)$. We prove (c): if $z = e^{i\theta}$ then

$$|B(e^{i\theta})| = \frac{|w_0 - e^{i\theta}|}{|1-\bar{w}_0 e^{i\theta}|} = \frac{|1-w_0 e^{-i\theta}|}{|1-\bar{w}_0 e^{i\theta}|} = 1,$$

thus B takes the boundary of the disc to itself. Thus, by the Maximum Modulus Principle we conclude that B takes values in $D_1(0)$.

Observe that B is injective: if $z_1 \neq z_2$ then

$$\frac{w_0 - z_1}{1 - \bar{w}_0 z_1} = \frac{w_0 - z_2}{1 - \bar{w}_0 z_2} \Leftrightarrow z_1 = z_2.$$

We can also show that B is surjective: let $\zeta \in D_1(0)$ then

$$\zeta = \frac{w_0 - z}{1 - \overline{w_0} z} \Leftrightarrow z = \frac{w_0 - \zeta}{1 - \zeta \overline{w_0}},$$

since $|\zeta| < 1$ and we conclude that $|z| < 1$ and B is surjective.

3. Let $U \subset \mathbf{C}$ be an open set containing the unit disc $\bar{D}_1(0)$. Let $f \in \mathcal{H}(U)$, and assume that f is not constant. Suppose further that $|f(z)| = 1$ if $|z| = 1$.

   (a) Show that there exists $z \in D_1(0)$ such that $f(z) = 0$.

   (b) Show that for any $w \in D_1(0)$, there exists z such that $f(z) = w$. (Hint: apply the previous question to an auxiliary function constructed using Exercise 2.)

   Solution:

   (a) First we observe that from the Maximum Modulus Principle it holds that $|f(z)| \leqslant 1$ for $z \in D_1(0)$. We suppose that f has no zeros in $D_1(0)$. Since $|f(z)| = 1$ for $|z| = 1$ we can take $\tilde{U}$ open set satisfying $f(z) \neq 0 \forall z \in D_1(0)$ and $\overline{D_1(0)} \subset \tilde{U}$. We define $G : \tilde{U} \to \mathbf{C}$, $G(z) = \frac{1}{f(z)}$. Observe that $|G(z)| = 1$ whenever $|z| = 1$ so, by the Maximum Modulus Principle it holds that $|G(z)| \leqslant 1$ for $z \in D_1(0)$, which is a contraction.

   (b) Let $U^\star$ be an open set such that $f(z) \neq \frac{1}{\overline{w}}$ and $\overline{D_1(0)} \subset U^\star$ (which exists since $|f(z)| = 1$ for $|z| = 1$). Define the function

   $$G : U^\star \to \mathbf{C}$$
   $$z \mapsto \frac{w - f(z)}{1 - \overline{w} f(z)}.$$

   From the previous exercise we can see that G maps $D_1(0)$ to $D_1(0)$ and that $|G(z)| = 1$ for $|z| = 1$. We can then conclude using the previous item that there exists z_0 such that $G(z_0) = 0 \Rightarrow f(z) = w$, as we wanted to show.

4. Let $r > 1$ be a real number and let f and g be functions holomorphic in $D_r(0)$. We assume that for $|z| \leqslant 1$, we have $f(z) = 0$ if and only if $z = 0$, and that $\mathrm{ord}_0(f) = 1$. We also assume that g is not the zero function.

   For $\varepsilon \in \mathbf{C}$, we denote
   $$f_\varepsilon(z) = f(z) + \varepsilon g(z).$$

(a) Show that there exists a real number $\delta > 0$ such that we have $|f(z)/g(z)| \geqslant \delta$ if z satisfies $|z| = 1$ and $g(z) \neq 0$.

(b) Show that if $|\varepsilon| < \delta$, the function f_ε is holomorphic on $D_r(0)$ and there is a unique z_ε such that $|z_\varepsilon| \leqslant 1$ and $f_\varepsilon(z_\varepsilon) = 0$. We denote by Z the map from $D_\delta(0)$ to $\bar{D}_1(0)$ such that $Z(\varepsilon) = z_\varepsilon$.

(c) Prove that for $|\varepsilon| < \delta$, we have $|z_\varepsilon| < 1$.

(d) Let (ε_n) be a sequence with $|\varepsilon_n| < \delta$ which converges to $\varepsilon \in D_\delta(0)$. If $Z(\varepsilon_n)$ converges to some complex number z, show that $z = Z(\varepsilon)$. (Hint: use the uniqueness of $Z(\varepsilon)$.)

(e) Show that Z is continuous. (Hint: use the following fact from analysis: if a bounded sequence (w_n) of complex numbers has the property that all convergent subsequences (w_{n_k}) have *the same limit* w, then (w_n) converges to w.)

Solution:

(a) Let $M = \sup_{|z|=1} |g(z)|$. Observe that there exists K such that $|f(z)| \geqslant K$ for $|z| = 1$ since we know that $f(z) \neq 0 \ \ \forall \ \ |z| = 1$. Thus, if $|z| = 1$ and $g(z) \neq 0$ we have
$$\left| \frac{f(z)}{g(z)} \right| \geqslant \frac{K}{M}.$$
We let $\delta := \frac{K}{M}$.

(b) Let $|\varepsilon| < \delta$. For $|z| = 1$ it holds that
$$|f(z) + \varepsilon g(z) - f(z)| = |\varepsilon g(z)| < \delta |g(z)| \leqslant |f(z)|,$$
so from Rouche's Theorem we conclude that f and f_ε have the same number of zeros in $D_1(0)$. Since f has only one zero with multiplicity 1, we conclude that there exists a unique zero z_ε, $f_\varepsilon(z_\varepsilon) = 0$.

(c) This follows directly from Rouche's theorem.

(d) Let $\varepsilon_n \to \varepsilon$ and suppose that $z_{\varepsilon_n} \to z$. Then, we know that
$$f(z_{\varepsilon_n}) + \varepsilon_n g(z_{\varepsilon_n}) = 0,$$
and when we let $n \to \infty$ we get
$$f(z) + \varepsilon(z) = 0,$$
observe that $|\varepsilon| < \delta$ and from uniqueness proved in item (b) we conclude that $z = Z(\varepsilon)$.

(e) To prove that Z is continuous we take $\varepsilon_n \to \varepsilon$, $|\varepsilon_n|, |\varepsilon| < \delta$. We know that $|Z(\varepsilon_n)| < 1$. Let $Z(\varepsilon_{n_k})$ be a convergent subsequence of $Z(\varepsilon_n)$. From the previous item we conclude that it has to converge to $Z(\varepsilon)$. Since this holds for every convergent subsequence of $Z(\varepsilon_n)$, which is a bounded sequence, we conclude that $Z(\varepsilon_n) \to Z(\varepsilon)$.