Exercise sheet 9

Exercise worth bonus points: Exercise 2

1. For each of the functions

$$f(z) = \exp(-\pi z^2), \qquad f(z) = \frac{1}{z^2+1},$$

show the following property: there exists $a > 0$ (not necessarily the same in both cases) such that:

(a) The function f is holomorphic in the open set $U_a \subset \mathbf{C}$ defined by

$$U_a = \{z \in \mathbf{C} \mid |\operatorname{Im}(z)| < a\}.$$

(b) There exists a real number $C \geqslant 0$ such that

$$|f(z)| \leqslant \frac{C}{1+x^2} \text{ for } z = x + iy \text{ with } |y| < a.$$

2. We consider a real number $a > 0$ and a function f which has the properties of Exercise 1 for this value of a. For $R > 0$ and $h \in \mathbf{R}$, we define

$$\hat{f}_R(h) = \int_{-R}^{R} f(x) e^{-2i\pi hx} dx.$$

(a) Show that the limit

$$\hat{f}(h) = \lim_{R \to +\infty} \hat{f}_R(h) = \int_{-\infty}^{+\infty} f(x) e^{-2i\pi hx} dx$$

exists.

(b) Let $b \in \mathbf{R}$ be such that $0 < b < a$. Show that for any $h \in \mathbf{R}$ we have

$$\lim_{R \to +\infty} \int_{\sigma_R} f(z) e^{-2i\pi hz} dz = 0, \qquad \lim_{R \to +\infty} \int_{\sigma_{-R}} f(z) e^{-2i\pi hz} dz = 0,$$

where σ_R is the vertical segment from R to $R - ib$.

(c) Let γ_R be the line segment from $R - ib$ to $-R - ib$, oriented from right to left. Show that

$$\hat{f}(h) = \lim_{R \to +\infty} \int_{\gamma_R} f(z) e^{-2i\pi h z} dz.$$

(d) Show that there exists $D \geqslant 0$ such that

$$\Big| \int_{\gamma_R} f(z) e^{-2i\pi h z} dz \Big| \leqslant D e^{-2\pi b h}$$

for all $h \in \mathbf{R}$, and then that $|\hat{f}(z)| \leqslant D e^{-2\pi b h}$.

(e) Show similarly that

$$\hat{f}(h) = \lim_{R \to +\infty} \int_{\gamma'_R} f(z) e^{-2i\pi h z} dz,$$

where γ'_R is the segment from $R + ib$ to $-R + ib$ oriented from right to left.

3. We continue with some $a > 0$ and a function f as in the previous exercise. Fix b such that $0 < b < a$. Define

$$g(z) = \frac{1}{e^{2i\pi z} - 1}.$$

(a) Show that $g \in \mathcal{M}(\mathbf{C})$ and has simple poles with residue $1/(2i\pi)$ at all $z \in \mathbf{Z}$, and no other pole.

(b) For $N \geqslant 1$, show that

$$\sum_{-N \leqslant n \leqslant N} f(n) = \int_{\Gamma_N} \frac{f(z)}{e^{2i\pi z} - 1} dz$$

where Γ_N is the boundary of the rectangle $[-N - 1/2, N + 1/2] \times [-b, b]$ oriented counterclockwise.

(c) Let Γ_N^+ be the segment from $N + 1/2 + ib$ to $-N - 1/2 + ib$ of Γ_N. Show that

$$\int_{\Gamma_N^+} \frac{f(z)}{e^{2i\pi z} - 1} dz = \sum_{k=0}^{+\infty} \int_{\Gamma_N^+} f(z) e^{-2i\pi (k+1) z} dz.$$

(d) Deduce that

$$\lim_{N \to +\infty} \int_{\gamma_N^+} \frac{f(z)}{e^{2i\pi z} - 1} dz = \sum_{k=0}^{+\infty} \hat{f}(k + 1),$$

where $\hat{f}$ is defined as in the previous exercise. (Hint: observe that the previous exercise implies that the series on the right-hand side converges absolutely so that the sum and the limit can be exchanged).

(e) Show similarly that

$$\lim_{N\to+\infty} \int_{\gamma_N^-} \frac{f(z)}{e^{2i\pi z}-1} dz = \sum_{k=0}^{+\infty} \hat{f}(-k),$$

where Γ_N^- is the segment from $N+1/2-ib$ to $-N-1/2-ib$.

(f) Conclude that

$$\sum_{n\in\mathbf{Z}} f(n) = \sum_{h\in\mathbf{Z}} \hat{f}(h)$$

("Poisson summation formula"); here both series are defined as the limit as $N\to+\infty$ of the sums from $-N$ to N.