1. a) Observe that $f(z) = e^{-\pi z^2}$ is homorphic in $\mathbb{C}$, so we oan take any $a \in \mathbb{R}$.

$f(z) = \dfrac{1}{z^2+1}$ has poles in $z = \pm i$ so we can take any $0 < a < 1$.

b) For $|y| < a$ we have

$$|e^{-\pi(x+iy)^2}| = e^{-\pi x^2} \cdot e^{\pi y^2} \leq e^{\pi a^2} \cdot \frac{C}{1+x^2}$$

since we know that $e^{-\pi x^2}$ decays paster that any polynomial.

We bound $\dfrac{1}{z^2+1}$ in parts:

① For $|x| \geq 1$:

$$|z^2+1| = \left((x^2-y^2+1)^2 + 4x^2y^2 \right)^{1/2}$$
$$\geq x^2$$

$$\Rightarrow \frac{1}{|z^2+1|} \leq \frac{1}{x^2} \leq 2 \frac{1}{x^2+1} .$$

② For $1/2 < |y| \leq 1$:

$$|z^2+1| \geq (x^2+3/4)$$

$$\Rightarrow \frac{1}{|z^2+1|} \leqslant \frac{1}{x^2+3/4} \leqslant 2 \cdot \frac{1}{x^2+1}$$

(3) For $|x| \leqslant 1$, $|y| \leqslant 1$ we know that f is bounded, so we can find $C' > 0$ s.t.

$$\frac{1}{|z^2+1|} \leqslant C' \leqslant C' \cdot \frac{1}{x^2+1} .$$

2. a) Observe that

$$\left| \int_{-R}^{R} f(x) \cdot e^{-2\pi i x a} \, dx \right| \leqslant \int_{-\infty}^{\infty} \frac{C}{x^2+1} < +\infty$$

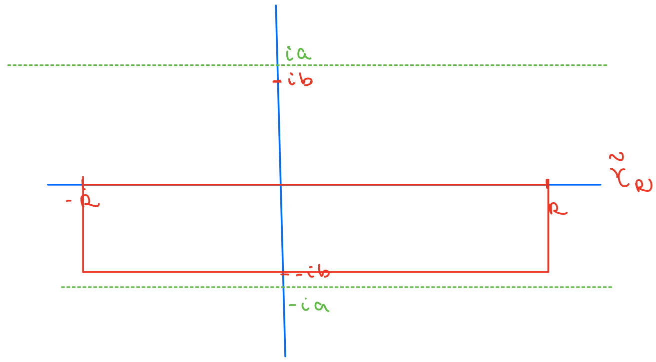
So we conclude that the limit exists.

b) $$\left| \int_{\sigma_R} f(z) \, dz \right| \leqslant C \int_{0}^{b} \frac{1}{R^2+1} \cdot e^{-2\pi a y} \, dx$$

$$\leqslant C(a) \frac{1}{R^2+1} \xrightarrow{R \to \infty} 0$$

The analogous holds for σ_{-R}.

c)



Observe that Cauchy's Thm implies that

$$\int_{\tilde{\gamma}_R} f(z)\cdot e^{-2\pi i a z}\,dz = 0$$

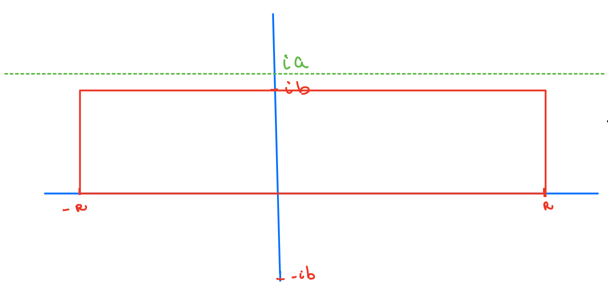
$$= -\int_{\sigma_R} f(z)\cdot e^{-2\pi i a z}\,dz + \int_{\sigma_R} f(z)\cdot e^{-2\pi i a z}\,dz - \int_{-R}^{R} f(x)\, e^{-2\pi i a x}\,dx + \int_{\gamma_R} f(z)\cdot e^{-2\pi i a z}\,dz$$

Taking the limit $R \longrightarrow \infty$ and using the previous item we get the result.

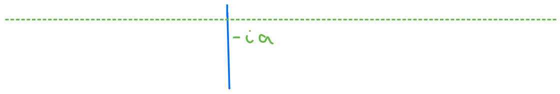
d) $$\left| \int_{\gamma_R} f(z)\, e^{-2\pi i a z}\,dz \right| \leq \int_{-R}^{R} \frac{C}{t^2+1}\cdot e^{-2\pi a b}\,dt$$

$$\leq \left(\int_{-\infty}^{\infty} \frac{C}{t^2+1}\,dt \right)\cdot e^{-2\pi a b} \leq \tilde{C}\cdot e^{-2\pi a b}$$

e) Observe that c) holds for the upper rectangle as well.



We prove the same result as in b for the segment R to $R+ib$, $\tilde{\sigma}_R$:



$$\left| \int_{\tilde{\sigma}_R} f(z)\,dz \right| \leq C \int_0^b \frac{1}{R^2+1} \cdot e^{2\pi a y} \leq \frac{C\, e^{2\pi a a} \cdot b}{R^2+1}$$

$$\leq C(a) \frac{1}{R^2+1} \xrightarrow{R \to \infty} 0$$

Thus the result holds.

3. a) Observe that the function has poles in $k \in \mathbb{Z}$. We compute the residues:

$$\frac{1}{e^{2\pi i z}-1} = \frac{1}{\cancel{1} + 2\pi i(z-k) - \cancel{1} + O((z-k)^2)}$$

$\Rightarrow$ the pole has order 1 and:

$$\lim_{z \to k} \frac{z-k}{2\pi i(z-k) + O((z-k)^2)} = \frac{1}{2\pi i}.$$

b) We use the Residue Thm:

$$\int_{\Gamma_N} \frac{f(z)}{e^{2\pi i}-1}\,dz = 2\pi i \sum_{|k| \leq N} \operatorname{Res}_k(g) = 2\pi i \sum_{|k|\leq N} \frac{f(k)}{2\pi i}$$

$$= \sum_{k=-N}^{N} f(k)$$

c) Observe that

$$\left|e^{2\pi i k z}\right| = \left|e^{2\pi i k(x+ib)}\right| = e^{-2\pi k b} \leq 1 \quad \text{thus}$$

$$-\sum_{k=0}^{\infty} e^{2\pi i k z} = \frac{1}{1-e^{2\pi i z}} \Rightarrow$$

$$-\int_{\Gamma_N^+} \frac{f(z)}{e^{2\pi i z}-1}\, dz = \int_{\Gamma_N^+} \sum_{k=0}^{\infty} f(z)\cdot e^{2\pi i k z}\, dz .$$

Since $\displaystyle\sum_{k=0}^{\infty} \left|f(z)e^{2\pi i k z}\right| \leq \sum_{k=0}^{\infty} \frac{c}{N^2+1}\cdot e^{-2\pi k b} < +\infty$

we can exchange the integral and the sum.

d) Since we can exchange the integral with the sum we get:

$$-\int_{\Gamma_N^+} \frac{f(z)}{e^{2\pi i z}-1}\, dz = \sum_{k=0}^{\infty} \hat{f}(-k) .$$

e) Analogously: $\quad Z = x - ib$

$$\sum_{k=0}^{\infty} e^{-2\pi i (k+1) z} = -\frac{1}{1-e^{-2\pi i z}}$$

$$\Longrightarrow \quad \lim_{N\to\infty} \int_{\Gamma_N^-} \frac{f(z)}{1-e^{-2\pi i z}}\, dz = \sum_{k=0}^{\infty} \int_{\Gamma_N^-} f(z)\, e^{-2\pi i (k+1) z}\, dz$$

d) The result follows from observing that

$$\left| \int_{-b}^{b} \frac{f(N+1/2+ib)}{e^{2\pi i (N+1/2+ix)}-1}\, dz \right| \leq \frac{2bC'}{(N+1/2)^2+1} \xrightarrow{N\to\infty} 0$$

and using what we proved above.