D-MATH | Complex Analysis | HS 2022
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Mock Exam

Exercise 1: multiple choice questions. There is exactly one correct answer for each question; correctly answered questions give one point, wrong answers or no answers give zero points (no negative points).

We always write $z = x + iy$ where x and y are the real and imaginary parts of z respectively.

a) Which of these functions $u\colon \mathbf{C} \to \mathbf{R}$ cannot be the real part of a holomorphic function $f\colon \mathbf{C} \to \mathbf{C}$:

I. $u(z) = x^2 + y^2$.

II. $u(z) = x^2 - y^2$.

III. $u(z) = e^x \cos(y)$.

IV. $u(z) = \cos(x)(e^y + e^{-y})$.

b) Which function is holomorphic on $\mathbf{C}$:

I. $f(z) = 1/z$

II. $f(z) = \mathrm{Re}(z)$

III. $f(z) = \exp(z^3)$

IV. $f(z) = \exp(\bar{z})$

c) Which of the following properties is *not* true for a holomorphic function $f\colon D_1(0) \to \mathbf{C}$:

I. $f(0) = \frac{1}{2\pi i} \int_{C_{1/2}(0)} \frac{f(w)}{w} dw$, where the circle is oriented counterclockwise.

II. f admits a power series expansion

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n$$

valid for $|z| < 1/2$.

III. f is bounded.

IV. $\int_{C_{1/2}(0)} f(z) dz = 0$, where the circle is oriented counterclockwise.

d) What is the value of
$$\int_\gamma \frac{e^z}{z^2 - 1/4} dz$$
where γ is the boundary of the rectangle $[0,1] \times [-1,1]$ taken counterclockwise:

I. 0.
II. $2ie\pi$.
III. $2i\pi e^{1/2}$.
IV. $e^{1/2}$.

e) What is the value of
$$\int_\gamma \frac{e^z}{z^2 - 1/4} dz$$
where γ is the boundary of the rectangle $[-1/4, 0] \times [-1,1]$ taken counterclockwise:

I. 0.
II. $2ie\pi$.
III. $2i\pi e^{1/2}$.
IV. $e^{1/2}$.

f) What is the residue of the function
$$f(z) = \frac{\cos(z)}{\sin(z)}$$
at $z = 2\pi$:

I. 0.
II. 1.
III. -1.
IV. π.

g) What is the value of
$$\frac{1}{2i\pi} \int_\gamma \frac{w^3 - w + 1}{(w-i)^2} dw,$$
where γ is the circle centered at $2i$ with radius 2 taken counterclockwise:

I. -2.
II. 2π.
III. -4.
IV. 4.

h) Which of the following properties is true for all holomorphic functions $f\colon \mathbf{C} \to \mathbf{C}$:

I. f is bounded.
II. there exists some integer $k \in \mathbf{Z}$ such that $f(k) \neq 0$.
III. the power series expansion of f around $2i$ has finite radius of convergence.
IV. if $f(z) = z$ for $|z| = 1$ then $f(z) = z$ for all $z \in \mathbf{C}$.

In the following exercises, please justify all steps.

Exercise 2. Find the power series expansions:

1. around 1 of the function
$$f(z) = \frac{1}{(1+z)^2}.$$
2. around 0 of the function
$$f(z) = e^{z^2}.$$

Exercise 3. Let $U = \{z \in \mathbf{C} \mid \mathrm{Re}(z) > 0\}$. Show that the integral

$$f(z) = \int_0^1 x^z(1-x)^z dx$$

exists for all $z \in U$ and that the function f defined in this way is holomorphic on U.

Exercise 4. Let U be the open set of all $z \in \mathbf{C}$ such that $e^z + 1 \neq 0$.

a) Determine the complement F of U in $\mathbf{C}$.

b) Let a be a complex number. Show that $f(z) = e^{az}/(1+e^z)$ is holomorphic on U.

c) For $z \in F$, show that f has a pole at z and compute its residue.

d) Let $R \geqslant 4$ be a real number and let γ_R be the boundary of the rectangle $[-R, R] \times [0, 2i\pi]$, taken counterclockwise. Show that

$$\int_{\gamma_R} f(z) dz = -2i\pi e^{i\pi a}.$$

What would happen if we took $R = 2$?

e) Let σ_1 be the vertical segment on the right of γ_R (which can be parameterized by $\sigma_1(t) = R + it$ for $0 \leqslant i \leqslant 2\pi$) and σ_2 be the vertical segment on the left. Show that if a is a real number with $0 < a < 1$, then

$$\lim_{R \to +\infty} \int_{\sigma_1} f(z)dz = \lim_{R \to +\infty} \int_{\sigma_2} f(z)dz = 0.$$

f) Show that the improper Riemann integral

$$\int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^x} dx$$

exists and is absolutely convergent for $0 < a < 1$ and that it is equal to

$$\frac{\pi}{\sin(\pi a)}.$$

g) What happens if we only assume that $0 < \mathrm{Re}(a) < 1$?