Mock Exam

Exercise 1: multiple choice questions. There is exactly one correct answer for each question; correctly answered questions give one point, wrong answers or no answers give zero points (no negative points).

We always write $z = x + iy$ where x and y are the real and imaginary parts of z respectively.

a) Which of these functions $u\colon \mathbf{C} \to \mathbf{R}$ cannot be the real part of a holomorphic function $f\colon \mathbf{C} \to \mathbf{C}$:

I. $u(z) = x^2 + y^2$.

II. $u(z) = x^2 - y^2$.

III. $u(z) = e^x \cos(y)$.

IV. $u(z) = \cos(x)(e^y + e^{-y})$.

Observe that if u is the real part of a holomorphic function then u has to be harmonic. But $\Delta(u) = 2 + 2 = 4 \neq 0$.

b) Which function is holomorphic on $\mathbf{C}$:

I. $f(z) = 1/z$

II. $f(z) = \mathrm{Re}(z)$

III. $f(z) = \exp(z^3)$

IV. $f(z) = \exp(\bar{z})$

c) Which of the following properties is *not* true for a holomorphic function $f\colon D_1(0) \to \mathbf{C}$:

I. $f(0) = \frac{1}{2\pi i} \int_{C_{1/2}(0)} \frac{f(w)}{w} dw$, where the circle is oriented counterclockwise.

II. f admits a power series expansion

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n$$

valid for $|z| < 1/2$.

III. f is bounded.

IV. $\int_{C_{1/2}(0)} f(z)dz = 0$, where the circle is oriented counterclockwise.

Consider $f(z) = \frac{1}{z-1}$.

d) What is the value of

$$\int_\gamma \frac{e^z}{z^2 - 1/4} dz$$

where γ is the boundary of the rectangle $[0, 1] \times [-1, 1]$ taken counterclockwise:

I. 0.

II. $2ie\pi$.

III. $2i\pi e^{1/2}$.

IV. $e^{1/2}$.

Cauchy's integral formula at $z = 1/2$ and $f(z) = \frac{e^z}{z+\frac{1}{2}}$.

e) What is the value of

$$\int_\gamma \frac{e^z}{z^2 - 1/4} dz$$

where γ is the boundary of the rectangle $[-1/4, 0] \times [-1, 1]$ taken counterclockwise:

I. 0.

II. $2ie\pi$.

III. $2i\pi e^{1/2}$.

IV. $e^{1/2}$.

Cauchy's Theorem.

f) What is the residue of the function

$$f(z) = \frac{\cos(z)}{\sin(z)}$$

at $z = 2\pi$:

I. 0.

II. 1.

III. -1.

IV. π.

Expand in Taylor series to note that $z = 2\pi$ is a pole of order 1.

g) What is the value of

$$\frac{1}{2i\pi}\int_\gamma \frac{w^3-w+1}{(w-i)^2}dw,$$

where γ is the circle centered at $2i$ with radius 2 taken counterclockwise:

I. -2.

II. 2π.

III. -4.

IV. 4.

Cauchy's integral formula for $f(z)=z^3-z+1$ at $z=1$.

h) Which of the following properties is true for all holomorphic functions $f\colon \mathbf{C}\to\mathbf{C}$:

I. f is bounded.

II. there exists some integer $k\in\mathbf{Z}$ such that $f(k)\neq 0$.

III. the power series expansion of f around $2i$ has finite radius of convergence.

IV. if $f(z)=z$ for $|z|=1$ then $f(z)=z$ for all $z\in\mathbf{C}$.

In the following exercises, please justify all steps.

Exercise 2. Find the power series expansions:

1. around 1 of the function
$$f(z)=\frac{1}{(1+z)^2}.$$

2. around 0 of the function
$$f(z)=e^{z^2}.$$

Solution:

1.
$$f(z)=\frac{1}{4}\sum_{n=1}^{\infty}(-1)^n n\frac{(z-1)^{n-1}}{2^{n-1}}.$$

2.
$$f(z)=\sum_{n=1}^{\infty}\frac{z^{2n}}{n!}.$$

Exercise 3. Let $U = \{z \in \mathbf{C} \mid \operatorname{Re}(z) > 0\}$. Show that the integral

$$f(z) = \int_0^1 x^z (1-x)^z dx$$

exists for all $z \in U$ and that the function f defined in this way is holomorphic on U.

Solution: Let $F : U \times [0,1] \to \mathbf{C}$, $F(z,x) = x^z(1-x)^z = e^{z(\log(x)+\log(1-x))}$ for $(z,x) \in U \times (0,1)$ and $F(z,0) = F(z,1) = 0$. We can see that F is is well defined and it is continuous for $(z,x) \in U \times (0,1)$. We check continuity for $U \times \{0,1\}$:

Let $z \in U$ and $x_n \to 0^+$. Observe that $\log(x_n) \to -\infty$ and since $\operatorname{Re}(z) > 0$ we get

$$\lim_{n\to\infty} |e^{z\log(x_n)}| = \lim_{n\to\infty} e^{\operatorname{Re} z \cdot \log(x_n)} = 0.$$

We prove that F is continuous in $U \times \{1\}$ analogously.

We observe that for any fixed $x \in [0,1]$, $F(z,x)$ is holomorphic in U, so by Theorem 5.4 we can conclude that f defines a holomorphic function in U.

Exercise 4. Let U be the open set of all $z \in \mathbf{C}$ such that $e^z + 1 \neq 0$.

a) Determine the complement F of U in $\mathbf{C}$.

b) Let a be a complex number. Show that $f(z) = e^{az}/(1+e^z)$ is holomorphic on U.

c) For $z \in F$, show that f has a pole at z and compute its residue.

d) Let $R \geqslant 4$ be a real number and let γ_R be the boundary of the rectangle $[-R,R] \times [0,2\pi]$, taken counterclockwise. Show that

$$\int_{\gamma_R} f(z)dz = -2i\pi e^{i\pi a}.$$

What would happen if we took $R = 2$?

e) Let σ_1 be the vertical segment on the right of γ_R (which can be parameterized by $\sigma_1(t) = R + it$ for $0 \leqslant t \leqslant 2\pi$) and σ_2 be the vertical segment on the left. Show that if a is a real number with $0 < a < 1$, then

$$\lim_{R\to+\infty} \int_{\sigma_1} f(z)dz = \lim_{R\to+\infty} \int_{\sigma_2} f(z)dz = 0.$$

f) Show that the improper Riemann integral

$$\int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^x} dx$$

exists and is absolutely convergent for $0 < a < 1$ and that it is equal to

$$\frac{\pi}{\sin(\pi a)}.$$

g) What happens if we only assume that $0 < \mathrm{Re}(a) < 1$?

Solution:

a) Observe that $F = \{z \in \mathbf{C} : e^z = -1\}$ and $e^z = -1 = e^{\pi i} \Leftrightarrow z = \pi i + 2\pi i k$ for $k \in \mathbf{Z}$, so

$$F = \{\pi i(2k+1), \text{ for } k \in \mathbf{Z}\}.$$

b) Observe that for $a \in \mathbf{C}$ e^{az} is holomorphic in $\mathbf{C}$ and, since we $e^z + 1$ is holormophic in U and never zero we conclude that f is holormorphic in U.

c) Let $z_k = \pi i(2k+1)$. We write

$$\frac{e^{az}}{1+e^z} = \frac{e^{az}}{-\sum_{n=1}^{\infty} \frac{(z-z_k)^n}{n!}} = \frac{e^{az}}{-(z-z_k)\left(1 + \sum_{n=2}^{\infty} \frac{(z-z_k)^{n-1}}{n!}\right)}$$

and observe that z_k is a pole of order 1 and

$$\mathrm{res}_{z_k}(f) = \lim_{z \to z_k} (z - z_k) \frac{e^{az}}{-(z-z_k)\left(1 + \sum_{n=2}^{\infty} \frac{(z-z_k)^{n-1}}{n!}\right)} = -e^{az_k}.$$

d) Let $R \geqslant 4$ and observe that the only singularity that lies inside the rectangle $[-R, R] \times [0, 2\pi]$ is $z = \pi i$, so by the Residue Theorem we have

$$\int_{\gamma_R} f(z)dz = 2\pi i \, \mathrm{res}_{\pi i}(f) = -2i\pi e^{i\pi a}.$$

When $R = 2$ the same result holds.

e) Observe that

$$\left| \int_0^{2\pi} \frac{e^{a(R+ix)}}{1+e^{R+ix}} dx \right| \leqslant 2\pi \frac{e^{aR}}{e^R - 1},$$

and since $0 < a < 1$, we get

$$\lim_{R \to +\infty} \int_{\sigma_1} f(z)dz = 0.$$

The same analysis holds for σ_2.

f) Using the results above we can write

$$\int_{-R}^{R} \frac{e^{ax}}{1+e^x} dx = \frac{1}{1-e^{2\pi i a}} \left(-2\pi i e^{a\pi i} + \int_{\sigma_1} f(z)dz + \int_{\sigma_2} f(z)dz \right),$$

so letting $R \to +\infty$ we get

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ax}}{1+e^x} dx = -\frac{1}{1-e^{2\pi i a}} 2\pi i e^{a\pi i} = \pi \left(\frac{-2i}{e^{-\pi i a} - e^{\pi i a}} \right) = \frac{\pi}{\sin(\pi a)}.$$

g) When $0 < \mathrm{Re}(z) < 1$ we observe that the result in e) still holds so we can conclude the same result in f) in this case.