Complex analysis
Summary of the main definitions/results

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General recommendations

- For every definition, have in mind (at least) one example and one counter-example.
- For every theorem (especially formulas), have in mind some examples.
- For every theorem, try to remember the *strategy* of the proof (for instance: *"Conformal maps have non-zero derivative comes from Rouché's Theorem"*).
- The statements in this summary are *not precise*. Check the notes!

Chapter I – Introduction

- Reminders on topology: **definitions** of open sets, closed sets, compact sets and connected sets.

Chapter II – Holomorphic functions

- **Definition**. Holomorphic functions.
- Rules for computing derivatives.
- Convergent power series define holomorphic functions.
- Cauchy–Riemann equations.
- **Definition**. (Smooth) curves, line integrals.
- **Definition**. Primitive.
- $\int_\gamma f(z)dz = F(z_2) - F(z_1)$ if $F' = f$.

Chapter III – Cauchy's Theorem

- A holomorphic function in a convex open set has a primitive.
- Cauchy's integral formula for a circle.

Chapter IV – Applications of Cauchy's Theorem

- $f \in \mathscr{H}(U)$ can be expanded in power series in any open disc contained in U.
- Cauchy's integral formula for $f^{(n)}(z_0)$.
- Liouville's Theorem.
- **Definition.** Order of vanishing of a holomorphic function at z_0.
- A non-constant holomorphic function on a connected open set has isolated zeros.
- Analytic continuation.
- Convergence theorem for sequences of holomorphic functions that converge locally uniformly.
- Holomorphic functions defined by integrals.

Chapter V – Meromorphic functions

- **Definition.** Poles of holomorphic functions, order of a pole.
- Removable singularities (if $f \in \mathscr{H}(U - \{z_0\})$ is bounded on a non-empty open punctured disc around z_0, then f has analytic continuation at z_0).
- **Definition**. Principal part and residue of a function at a pole.
- The residue formula for a circle.
- **Definition.** Meromorphic functions, $\widehat{\mathbf{C}}$.
- $\frac{1}{2i\pi}\int_\gamma f'/f =$ number of zeros with multiplicity inside $\gamma -$ number of poles with multiplicity inside γ.
- Rouché's Theorem.
- Open Image Theorem.
- Maximum Modulus Principle.

Chapter VI – Eta, THeta, Zeta

- **Definition.** Infinite products of complex numbers.
- If $\sum |a_n(z)|$ converges locally uniformly, then $\prod(1 + a_n(z))$ is holomorphic.

Chapter VII – Homotopy and applications

- **Definition.** Homotopy of smooth curves.
- Homotopy Theorem.
- **Definition.** Simply connected open set.
- If $f \in \mathscr{H}(U)$, U simply connected open set, then f has a primitive.
- Cauchy's formula in a simply connected open set.
- **Definition.** Branch of the logarithm on some open set; principal branch of the logarithm.
- If U is simply connected, there is a branch of the logarithm on U.
- **Definition.** z^α for z in a simply connected open set; n-th root of z.
- **Definition.** Winding number of a curve around a point.
- Residue formula with winding numbers for $f \in \mathscr{M}(U)$, U simply connected.

Chapter VIII – Conformal mapping

- **Definition.** Conformal map; conformal equivalence; automorphism.
- Conformal maps have non-zero derivative everywhere.
- Example of conformal maps $\mathbf{H} \to D_1(0)$.
- Riemann's Mapping Theorem.
- Automorphisms of $D_1(0)$.
- Schwarz's Lemma.
- If (f_n) is a sequence of injective holomorphic functions which converges locally uniformly, then the limit is constant or injective.
- Montel's Theorem.