AT I - Solution

1 a) Let $\mathcal{U} \subset X$, $V \subset Y$ be open nbhds such that x_0 is a deformation retract of $\mathcal{U}$ and y_0 is a defo. retr. of V.

Set $A := X \cup V$, $B := \mathcal{U} \cup Y \subsetneq X \vee Y$. Then $A \cup B = X \vee Y$ and

A deformation retracts onto $X \subset X \vee Y$ and B deform. retracts onto $Y \subset X \vee Y$ and $A \cap B$ deform. retracts onto $p \in X \vee Y$.

(relative) M-V-sequence:

$$\cdots \to \underbrace{H_n(p,p)}_{=0} \to H_n(X,x_0) \oplus H_n(Y,y_0) \longrightarrow H_n(X \vee Y, p) \to \underbrace{H_{n-1}(p,p)}_{=0} \to$$

$$\Rightarrow H_n(X \vee Y, p) \cong H_n(X,x_0) \oplus H_n(Y,y_0) \quad \forall n. \quad \square$$

b) $\mathbb{RP}^\infty$ has a CW structure with one cell e^k in each dimension $k \geq 0$.
The attaching map for e^k is the 2-sheeted covering projection $\varphi: S^{k-1} \to \mathbb{RP}^{k-1}$. To compute the boundary d_k we need to compute the degree of the composition $S^{k-1} \xrightarrow{\varphi} \mathbb{RP}^{k-1} \xrightarrow{q} \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \cong S^{k-1}$

q: quotient

$q \circ \varphi$ is a homeomorphism when restricted to each component of $S^{k-1} \setminus S^{k-2}$ and these two homeomorphisms differ by the antipodal map of S^{k-1} which has degree $(-1)^k$ $\Rightarrow \deg(q \circ \varphi) = 1 + (-1)^k = \begin{cases} 2, & k = \text{even} \\ 0, & k = \text{odd} \end{cases}$

The cellular chain complex of $\mathbb{RP}^\infty$ is:

$$\cdots \to \mathbb{Z} \xrightarrow[\cdot 2]{d_{2k}} \mathbb{Z} \xrightarrow[\cdot 0]{d_{2k-1}} \mathbb{Z} \xrightarrow{\cdot 2} \cdots \xrightarrow[0]{d_1} \mathbb{Z} \to 0$$

$$\Rightarrow H_n^{CW}(\mathbb{RP}^\infty) \cong \begin{cases} \mathbb{Z}_2, & n = \text{odd}, n > 0 \\ \mathbb{Z}, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

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2a) $f(x) \neq x \;\; \forall x \in S^n \Rightarrow$ the line segment $(1-t)f(x) - t\cdot x$, $t \in [0,1]$

does not pass through 0.

Hence, if f has no fixed points, $f_t(x) := \dfrac{(1-t)f(x) - t\cdot x}{\|(1-t)f(x) - t\cdot x\|}$

is a well-defined homotopy with $f_0 = f$ and $f_1 = -id : S^n \circlearrowleft$

$\Rightarrow \deg(f) = \deg(-id) = (-1)^{n+1}$. $\square$

b) $g: S^n \to S^n$ with $\deg(g) = 0$. By 2a): $\deg(g) \neq (-1)^{n+1} \Rightarrow \exists x \in S^n$ with $g(x) = x$.

By 2a) again: $\deg(-id_{S^n} \circ g) = \deg(-id)\cdot\underbrace{\deg(g)}_{0} = 0 \neq (-1)^{n+1}$

$\Rightarrow \exists y \in S^n$: $\underbrace{-id \circ g(y)}_{-g(y)} = y$ hence $g(y) = -y$ $\square$

c) The degree of a homeomorphism is ± 1. Hence an action of G on S^n determines a degree function $d: G \to \{\pm 1\}$. This is a homomorphism, since $\deg(f\circ g) = \deg(f)\cdot\deg(g)$. If the action is free, then d maps every $\alpha \in G \setminus \{0\}$ to $(-1)^{n+1}$ by problem 2a) (since the corresp. homeom. has no fixed points).

$\Rightarrow$ If $n =$ even $\Rightarrow \ker d = 0 \Rightarrow G < \mathbb{Z}_2$. $\square$

ATI - Solution

3a) Let $f: \mathbb{RP}^2 \to \mathbb{RP}^2$ be a map that induces an isomorphism on homology. Show that f is surjective.

<u>Proof</u>: Assume that f is <u>not</u> surjective. Let $x_0 \in \mathbb{RP}^2 \setminus \mathrm{im}(f)$.

Then f factors as:

$$\begin{tikzcd} \mathbb{RP}^2 \arrow[r] \arrow[rr, bend right, "f"'] & \mathbb{RP}^2 \setminus \{x_0\} \arrow[r, hook, "i"] & \mathbb{RP}^2 \end{tikzcd}$$

$$\mathbb{RP}^2 \setminus \{pt\} \underset{\text{homotopy equivalent}}{\simeq} S^1 \cong \mathbb{RP}^1$$

We get:

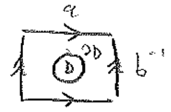
$$\begin{tikzcd} \mathbb{Z}_2 \cong H_1(\mathbb{RP}^2) \arrow[r] \arrow[rr, bend right, "f_*"'] & H_1(\mathbb{RP}^2 \setminus \{x_0\}) \cong \mathbb{Z} \arrow[r, "i_*"] & H_1(\mathbb{RP}^2) \cong \mathbb{Z}_2 \end{tikzcd}$$

Any homomorphism $\mathbb{Z}_2 \to \mathbb{Z}$ is trivial $\Rightarrow f_* = 0$

Hence f_* can't be an isomorphism on $H_1(\mathbb{RP}^2)$. $\square$

ATI – Solution

3b) T^2: torus, $D \subset T^2$ embedded disk; $X := T^2 \setminus \mathring{D}$



Prove that X doesn't retract onto $\partial D \subset X$.

<u>Proof</u>: Suppose there is a retraction $X \xrightarrow{r} \partial D$.

Let $i : \partial D \hookrightarrow X$ be the inclusion, then we have: $r \circ i = id_{\partial D}$. Hence:

$$
\begin{tikzcd}
\mathbb{Z} \cong H_1(\partial D) \arrow[r, "i_*"] \arrow[rr, bend right, "\cong"', "id_*"'] & H_1(X) \arrow[r, "r_*"] & H_1(\partial D) \cong \mathbb{Z}
\end{tikzcd}
$$

Let $\gamma \in \pi_1(\partial D)$ be a generator. $\pi_1(X) \cong \pi_1(\infty) = \langle a, b \rangle$ ← free group on 2 generators

$H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$.

now $i_\#(\gamma)$ gets mapped to $aba^{-1}b^{-1} \in \pi_1(X)$ i.e. to an element in the commutator. $\Longrightarrow i_* : H_1(\partial D) \to H_1(X)$ is the zero map.

$\Longrightarrow id_* = 0$ ↯ $\square$

Exercise 1.4.4 Consider the homology $H_*(C)$ of C as a chain complex with zero differentials. Show that if the complex C is split, then there is a chain homotopy equivalence between C and $H_*(C)$. Give an example in which the converse fails.

Exercise 1.4.5 In this exercise we shall show that the chain homotopy classes of maps form a quotient category **K** of the category **Ch** of all chain complexes. The homology functors H_n on **Ch** will factor through the quotient functor **Ch** $\to$ **K**.

1. Show that chain homotopy equivalence is an equivalence relation on the set of all chain maps from C to D. Let $\mathrm{Hom}_{\mathbf{K}}(C, D)$ denote the equivalence classes of such maps. Show that $\mathrm{Hom}_{\mathbf{K}}(C, D)$ is an abelian group.
2. Let f and g be chain homotopic maps from C to D. If $u: B \to C$ and $v: D \to E$ are chain maps, show that vfu and vgu are chain homotopic. Deduce that there is a category **K** whose objects are chain complexes and whose morphisms are given in (1).
3. Let f_0, f_1, g_0, and g_1 be chain maps from C to D such that f_i is chain homotopic to g_i $(i = 1, 2)$. Show that $f_0 + f_1$ is chain homotopic to $g_0 + g_1$. Deduce that **K** is an additive category, and that **Ch** $\to$ **K** is an additive functor.
4. Is **K** an abelian category? Explain.

1.5 Mapping Cones and Cylinders

5)

1.5.1 Let $f: B_. \to C_.$ be a map of chain complexes. The *mapping cone* of f is the chain complex cone(f) whose degree n part is $B_{n-1} \oplus C_n$. In order to match other sign conventions, the differential in cone(f) is given by the formula

$$d(b, c) = (-d(b), d(c) - f(b)), \quad (b \in B_{n-1}, c \in C_n).$$

That is, the differential is given by the matrix

$$\begin{bmatrix} -d_B & 0 \\ -f & +d_C \end{bmatrix} : \begin{array}{ccc} B_{n-1} & \xrightarrow{-} & B_{n-2} \\ \oplus & \searrow^{-} & \oplus \\ C_n & \xrightarrow[+]{} & C_{n-1} \end{array}.$$

Here is the dual notion for a map $f: B^{\cdot} \to C^{\cdot}$ of cochain complexes. The mapping cone, cone(f), is a cochain complex whose degree n part is $B^{n+1} \oplus C^n$. The differential is given by the same formula as above with the same signs.

Exercise 1.5.1 Let cone(C) denote the mapping cone of the identity map id_C of C; it has $C_{n-1} \oplus C_n$ in degree n. Show that cone(C) is split exact, with $s(b, c) = (-c, 0)$ defining the splitting map.

Exercise 1.5.2 Let $f: C \to D$ be a map of complexes. Show that f is null homotopic if and only if f extends to a map $(-s, f): \mathrm{cone}(C) \to D$.

5a)

1.5.2 Any map $f_*: H_*(B) \to H_*(C)$ can be fit into a long exact sequence of homology groups by use of the following device. There is a short exact sequence

$$0 \to C \to \mathrm{cone}(f) \xrightarrow{\delta} B[-1] \to 0$$

of chain complexes, where the left map sends c to $(0, c)$, and the right map sends (b, c) to $-b$. Recalling (1.2.8) that $H_{n+1}(B[-1]) \cong H_n(B)$, the homology long exact sequence (with connecting homomorphism ∂) becomes

$$\cdots \to H_{n+1}(\mathrm{cone}(f)) \xrightarrow{\delta*} H_n(B) \xrightarrow{\partial} H_n(C) \to H_n(\mathrm{cone}(f)) \xrightarrow{\delta*} H_{n-1}(B) \xrightarrow{\partial} \cdots.$$

The following lemma shows that $\partial = f_*$, fitting f_* into a long exact sequence.

b)

Lemma 1.5.3 *The map ∂ in the above sequence is f_*.*

Proof If $b \in B_n$ is a cycle, the element $(-b, 0)$ in the cone complex lifts b via δ. Applying the differential we get $(db, fb) = (0, fb)$. This shows that

$$\partial[b] = [fb] = f_*[b]. \qquad \diamond$$

c)

Corollary 1.5.4 *A map $f: B \to C$ is a quasi-isomorphism if and only if the mapping cone complex cone(f) is exact. This device reduces questions about quasi-isomorphisms to the study of split complexes.*

Topological Remark Let K be a simplicial complex (or more generally a cell complex). The *topological cone* CK of K is obtained by adding a new vertex s to K and "coning off" the simplices (cells) to get a new $(n+1)$-simplex for every old n-simplex of K. (See Figure 1.1.) The simplicial (cellular) chain complex $C_.(s)$ of the one-point space $\{s\}$ is R in degree 0 and zero elsewhere. $C_.(s)$ is a subcomplex of the simplicial (cellular) chain complex $C_.(CK)$ of

13. Euler's Formula

Let us recall, without proof, the Fundamental Theorem of Abelian Groups. A finitely generated free abelian group A is isomorphic to $\mathbf{Z}^r$ for some r. Suppose that $B \subset A$ is a subgroup. Then there exists a basis $a_1, \ldots, a_r$ of A and nonzero integers $n_1 | n_2 | \cdots | n_s$ (each dividing the next) with $s \le r$ such that $n_1 a_1, \ldots, n_s a_s$ is a basis for B. In particular, B is free abelian of rank s and

$$A/B \approx \mathbf{Z}_{n_1} \oplus \cdots \oplus \mathbf{Z}_{n_s} \oplus \mathbf{Z}^{r-s}. \tag{1}$$

The integer $r - s \ge 0$ is called the rank of A/B. Note that it is the dimension of the rational vector space $(A/B) \otimes \mathbf{Q}$ where $\mathbf{Q}$ is the rationals.

Thus any finitely generated abelian group has the form of (1) and if $0 \to B \to A \to C \to 0$ is an exact sequence of finitely generated abelian groups then $\operatorname{rank}(A) = \operatorname{rank}(B) + \operatorname{rank}(C)$.

13.1. Definition. A space X is said to be of *finite type* if $H_i(X)$ is finitely generated for each i. It is of *bounded finite type* if $H_i(X)$ is also zero for all but a finite number of i.

13.2. Definition. If X is a space of bounded finite type then its *Euler characteristic* is

$$\chi(X) = \sum_i (-1)^i \operatorname{rank} H_i(X).$$

Note then that $\chi(X)$ is a topological invariant of X.

4a) **13.3. Theorem** (Euler–Poincaré). *Let X be a finite CW-complex and let a_i be the number of i-cells in X. Then $\chi(X)$ is defined and*

$$\chi(X) = \sum_i (-1)^i a_i.$$

Proof. Note that $a_i = \operatorname{rank} C_i(X)$. Let $Z_i \subset C_i = C_i(X)$ be the group of i-cycles, $B_i = \partial C_{i+1}$, the group of i-boundaries, and $H_i = H_i(C_*(X)) \approx H_i(X)$. Thus $H_i = Z_i/B_i$.

The exact sequence

$$0 \to Z_i \to C_i \xrightarrow{\partial} B_{i-1} \to 0$$

shows that

$$\operatorname{rank}(C_i) = \operatorname{rank}(Z_i) + \operatorname{rank}(B_{i-1}).$$

Similarly the exact sequence $0 \to B_i \to Z_i \to H_i \to 0$ shows that

$$\operatorname{rank}(Z_i) = \operatorname{rank}(B_i) + \operatorname{rank}(H_i).$$

Adding the last two equations with signs $(-1)^i$ gives

$$\sum_i (-1)^i (\operatorname{rank}(H_i) + \operatorname{rank}(B_i)) = \sum_i (-1)^i \operatorname{rank}(Z_i)$$

$$= \sum_i (-1)^i (\operatorname{rank}(C_i) - \operatorname{rank}(B_{i-1})).$$

The terms in B_* cancel, leaving $\chi(X) = \sum_i (-1)^i \operatorname{rank}(H_i) = \sum_i (-1)^i \operatorname{rank}(C_i) = \sum_i (-1)^i a_i$. □

13.4. Corollary (Euler). *For any CW-complex structure on the 2-sphere with F 2-cells, E 1-cells and V 0-cells, we have $F - E + V = 2$.* □

4b) **13.5. Proposition.** *If $X \to Y$ is a covering map with k sheets (k finite) and Y is a finite CW-complex then X is also a CW-complex and $\chi(X) = k\chi(Y)$.*

PROOF. Since the characteristic maps $\mathbf{D}^i \to Y$ are maps from a simply connected space, they lift to X in exactly k ways. This gives the structure of a CW-complex on X with the number of i-cells exactly k times that number for Y. (Also see Theorem 8.10.) Thus the alternating sum of these for X is k times the same thing for Y. □

13.6. Corollary. *If $\mathbf{S}^{2n} \to Y$ is a covering map and Y is CW then the number of sheets is either* 1 *or* 2. □

13.7. Corollary. *The Euler characteristic of real projective $2n$-space $\mathbf{P}^{2n}$ is* 1. □

13.8. Corollary. *If $f: \mathbf{P}^{2n} \to Y$ is a covering map and Y is a CW-complex then f is a homeomorphism.* □

The hypothesis that Y is a CW-complex in Corollaries 13.6 and 13.8 can be dropped, but we do not now have the machinery to prove that.

PROBLEMS

1. Use the knowledge of the covering spaces of the torus, but do not use the knowledge of its homology groups, to show that its Euler characteristic is zero.

2. If X is a finite CW-complex of dimension two, and if X is simply connected then show that $\chi(X)$ determines $H_2(X)$ completely. What are the possible values for $\chi(X)$ in this situation?

3. Let

$$A(t) = \sum_{i=0}^{\infty} a_i t^i \quad \text{and} \quad B(t) = \sum_{i=0}^{\infty} b_i t^i$$

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6) Compute $\tilde{H}_i(S^n \setminus X)$, $X \subsetneq S^n$ and

a) $X \approx S^k \vee S^l$, $0<k,l<n$

b) $X \approx S^k \sqcup S^l$, $0<k,l<n$

Solution:

Let $K_1, K_2 \subset S^n$ be closed subspaces and $A_j = S^n \setminus K_j$, $j=1,2$

and $Y := A_1 \cup A_2 \subset S^n$. Then $A_1, A_2 \subset Y$ is an open cover of Y

Let $K_1 \approx S^k$, $K_2 \approx S^l$

a) Suppose $K_1 \cap K_2 = \{x_0\} \subset S^n$. $A_1 \cap A_2 \neq \emptyset$ Reduced Mayer-Vietoris:

$$\ldots \to \tilde{H}_i(\underbrace{A_1 \cap A_2}_{\approx S^n \setminus S^k \vee S^l}) \to \underset{S^n \setminus S^k}{\tilde{H}_i(A_1)} \oplus \underset{S^n \setminus S^l}{\tilde{H}_i(A_2)} \to \underbrace{\tilde{H}_i(\underset{S^n \setminus \{x_0\} \approx \mathbb{R}^n}{Y})}_{=0} \to \tilde{H}_{i-1}(A_1 \cap A_2) \to \ldots$$

By the generalized Jordan thm. we get $\tilde{H}_i(S^n \setminus S^k) \cong \tilde{H}_i(S^{n-k-1}) \cong \begin{cases} \mathbb{Z}, & i=n-k-1 \\ 0, & \text{else} \end{cases}$

$$\Rightarrow \tilde{H}_i(S^n \setminus S^k \vee S^l) \cong \tilde{H}_i(S^n \setminus S^k) \oplus \tilde{H}_i(S^n \setminus S^l)$$

$$\cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & i=n-k-1,\ k=l \\ \mathbb{Z}, & i=n-k-1,\ k \neq l \\ \mathbb{Z}, & i=n-l-1,\ k \neq l \\ 0, & \text{else} \end{cases}$$

b) Same approach except that now: $K_1 \cap K_2 = \emptyset$

$$\ldots \to \tilde{H}_i(\underbrace{A_1 \cap A_2}_{\approx S^n \setminus (S^k \sqcup S^l)}) \to \underset{S^n \setminus S^k}{\tilde{H}_i(A_1)} \oplus \underset{S^n \setminus S^k}{\tilde{H}_i(A_2)} \to \underbrace{\tilde{H}_i(\underset{S^n}{Y})}_{\cong \begin{cases} \mathbb{Z}, & i=n \\ 0, & \text{else} \end{cases}} \to \tilde{H}_{i-1}(A_1 \cap A_2) \to \ldots$$

For $i=n$: $\tilde{H}_n(S^n \setminus S^k) \oplus \tilde{H}_n(S^n \setminus S^l) = 0$ since $n \neq n-k-1$ $\forall 0<k<n$ similarly $n \neq n-l-1$ $\forall 0<l<n$

$$\Rightarrow \tilde{H}_n(S^n \setminus S^k \sqcup S^l) = 0$$

For $i=n-1$: $0 \to \tilde{H}_n(Y) \cong \mathbb{Z} \to \tilde{H}_{n-1}(S^n \setminus S^k \sqcup S^l) \to \underbrace{\tilde{H}_{n-1}(S^n \setminus S^k)}_{=0 \text{ since } n-1 \neq n-k-1} \oplus \underbrace{H_{n-1}(S^k \setminus S^l)}_{=0 \text{ since } n-1 \neq n-l-1\ \forall 0<k,l<n} \to \ldots$

$$\Rightarrow \tilde{H}_{n-1}(S^n \setminus S^k \sqcup S^l) \cong \mathbb{Z}$$

$$\Rightarrow \tilde{H}_i(S^n \setminus S^k \sqcup S^l) \cong \begin{cases} \mathbb{Z}, & i=n-1,\ 0<k,l<n \\ \tilde{H}_i(S^n \setminus S^k \vee S^l), & i \neq n-1 \end{cases}$$