A1 a) Suppose by contradiction that $\exists\, p \in \mathrm{Int}(K)$ s.t. $p \notin \bigcup_{i=1}^{e} \mathrm{Im}(\sigma_i)$

As X has only one n-dim cell $X \setminus \{p\}$ is homotopy equivalent to the $n-1$ skeleton $X^{(n-1)}$.

Then each σ_i can be factored as

$$\begin{tikzcd} \Delta^n \arrow[r, "\sigma_i'"] \arrow[rr, bend right, "x"'] & X \setminus \{p\} \arrow[r, hook, "i"] & X \end{tikzcd}.$$

Let $c' := \sum_i n_i \sigma_i' \in \underset{\text{cycles}}{S_n}(X \setminus \{p\})$

(c' is a cycle bc. c is one.)

Then

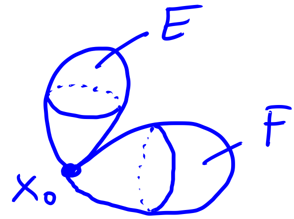
$[c'] \in H_n(X \setminus \{p\}) \cong H_n(X^{(n-1)}) = 0$,

so $[c'] = 0$

and $[c] = i_*([c']) = i_*(0) = 0$

$\square$

b) No, it is not true.



Counterexample

$X = S^n \vee S^n$

& $c : \Delta^n \longrightarrow S^n \vee S^n$ maps homeomorphically onto $\mathrm{Int}(E)$ & $\partial(\Delta^n)$ maps to x_0.

Clearly, c is a cycle and

$0 \neq [c] \in H_n(S^n \vee S^n) \cong \mathbb{Z} \oplus \mathbb{Z}$

But $\mathrm{int}(F) \not\subseteq \mathrm{Im}(c)$

$\square$

A2 a) Let $C_n(M)$ denote the free abelian group generated by the n-cells of M.
Then

$C_0(M) \cong \mathbb{Z}p \oplus \mathbb{Z}q$

$C_1(M) \cong \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \oplus \mathbb{Z}\gamma$

$C_2(M) \cong \mathbb{Z} \cdot A$

$C_i(M) = 0 \quad \forall i \neq 0,1,2$

and the differential

$d_n : C_i \longrightarrow C_{i-1}$ can be computed using the cellular boundary formula.

$$d_n(\sigma) = \sum_{\tau \in C_{n-1}} [\tau : \sigma]\, \tau$$

$[\tau : \sigma] := \deg(p_\tau \circ f_{\partial\sigma})$, where

$f_{\partial\sigma}$ is the restriction of the characteristic map of σ to $\partial\sigma$ and

p_τ is the map that collapses $X^{(n-1)} \setminus \tau \subseteq X^{(n-1)}$ to a point.

The chain complex looks like

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

$$A \longmapsto dA = \alpha - \gamma + \alpha + \beta$$
$$= 2\alpha + \beta - \gamma$$
$$\alpha \longmapsto d\alpha = q - p$$
$$\beta \longmapsto d\beta = p - q$$
$$\gamma \longmapsto d\gamma = q - p$$

Hence:

$\operatorname{Ker} d_0 \cong \mathbb{Z}p \oplus \mathbb{Z}q$, $\operatorname{Im} d_1 \cong \mathbb{Z}(p-q)$

$\Longrightarrow H_0^{CW}(M) \cong \mathbb{Z}$

$\operatorname{Ker} d_1 \cong \mathbb{Z}(\alpha - \beta) \oplus \mathbb{Z}(\alpha - \gamma)$

$\cong \mathbb{Z}(2\alpha + \beta - \gamma) \oplus \mathbb{Z}(\alpha - \gamma)$

$\operatorname{Im} d_2 \cong \mathbb{Z}(2\alpha + \beta - \gamma)$

$\Longrightarrow H_1^{CW}(M) \cong \mathbb{Z}$

$\operatorname{Ker} d_2 = 0 \Longrightarrow H_2^{CW}(M) = 0$

b) $i_*: H_1^{CW}(\partial M) \longrightarrow H_1^{CW}(M)$ maps the generator $[\beta+\gamma] \in H_1^{CW}(\partial M)$ to $[2\alpha+\beta-\gamma] - 2[\alpha-\gamma]$

$= -2 \cdot$ generator of $H_1^{CW}(M)$

Hence this map is multiplication by 2 (or by -2).

c) <u>Solution 1:</u>

$C_2(M,\partial M) \cong \mathbb{Z}\cdot A$ $\qquad$ $C_i(M,\partial M)=0$ $\forall i \neq 1,2$

$C_1(M,\partial M) \cong \mathbb{Z}\alpha$

$C_0(M,\partial M) = 0$

chain complex:

$$
\begin{array}{lccccccccc}
\text{degree:} & & & 2 & & 1 & & 0 & & \\
& 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\
& & & A & \longmapsto & dA = 2\alpha & & & & \\
& & & & & \alpha & \longmapsto & 0 & &
\end{array}
$$

$\rightarrow H_2^{CW}(M,\partial M) = 0$

$H_1^{CW}(M,\partial M) \cong \mathbb{Z}/2\mathbb{Z}$

$H_0^{CW}(M,\partial M) = 0$

c) solution 2:

We could also use the l.e.s. of the pair $(M, \partial M)$:

$$0 \to H_2(\partial M) \to \underset{\overset{\|}{0}}{H_2(M)} \to H_2(M, \partial M) \to$$

$$\underset{\overset{\cong}{\mathbb{Z}}}{H_1(\partial M)} \xrightarrow{\times 2} \underset{\overset{\cong}{\mathbb{Z}}}{H_1(M)} \longrightarrow H_1(M, \partial M) \longrightarrow$$

$$\underset{\overset{\cong}{\mathbb{Z}}}{H_0(\partial M)} \hookrightarrow \underset{\overset{\cong}{\mathbb{Z}}}{H_0(M)} \longrightarrow 0$$

$H_0(\partial M) \longrightarrow H_0(M)$ is injective (and hence an iso),

as ∂M & M are both path-connected.

Also by b) the map $H_1(\partial M) \xrightarrow{\times 2} H_1(M)$ is injective.

Hence $H_2(M, \partial M) \cong \ker(H_1(\partial M) \to H_1(M)) = 0$

and $H_1(M, \partial M) \cong H_1(M) / \mathrm{Im}(H_1(\partial M) \to H_1(M)) \cong \mathbb{Z}/2\mathbb{Z}$.

A3 a) $f \times \mathrm{Id} : X \times I \longrightarrow Y \times I$ is continuous.

$$(x,t) \longmapsto (f(x), t)$$

As $f \times \mathrm{Id}(X \times \{0\}) \subseteq Y \times \{0\}$ & similar for $X \times \{1\}$, this descends to a continuous map $\Sigma f : \Sigma X \longrightarrow \Sigma Y$ and the diagram

$$\begin{tikzcd} X \times I \arrow[r, "f \times \mathrm{Id}"] \arrow[d, "q_X"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \times I \arrow[d, "q_Y"] \\ \Sigma X \arrow[r, "\Sigma f"] & \Sigma Y \end{tikzcd}$$

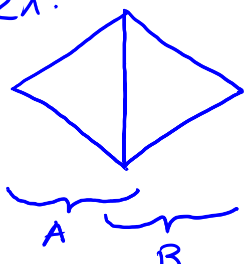
commutes.

q_Y and $f \times \mathrm{Id}$ are continuous, hence also $q_Y \circ f \times \mathrm{Id} : X \times I \longrightarrow \Sigma Y$ is continuous. By the universal property of the quotient space there exists a unique continuous map $g : \Sigma X \longrightarrow \Sigma Y$ s.t.

$$g \circ q_X = q_Y \circ f \times \mathrm{Id}.$$

Uniqueness of g implies that $\Sigma f = g$ and hence Σf must be continuous.

b) ΣX:



let $A := q_X(X \times [0, \frac{3}{4}))$
$B := q_X(X \times (\frac{1}{2}, 1])$

$\Sigma X = A \cup B$,
A, B are open

$A \cap B = q_X(X \times (\frac{1}{4}, \frac{3}{4}))$ deformation retracts onto $X \times \{\frac{1}{2}\}$
A and B are contractible.
(Cones are contractible as we saw in an exercise.)
We can apply reduced Mayer Vietoris:

$$\begin{tikzcd}[column sep=small]
\tilde H_{n+1}(A) \oplus \tilde H_{n+1}(B) \arrow[r] & \tilde H_{n+1}(A \cup B) \arrow[r, "\partial_*"] \arrow[d, phantom, "\shortparallel"] & \tilde H_n(A \cap B) \arrow[r] \arrow[d, phantom, "\cong"] & H_n(A) \oplus H_n(B) \\
0 \quad\quad 0 & \tilde H_{n+1}(\Sigma X) & \tilde H_n(X) & 0 \quad\quad 0
\end{tikzcd}$$

$\Rightarrow \partial_*$ induces an isomorphism:

$$\tilde H_{n+1}(\Sigma X) \cong \tilde H_n(X) \quad \forall n \qquad \square$$

b) (continued)

Naturality follows directly from the naturality of the Mayer Vietoris sequence.

I.e. $\forall$ continuous map $f: X \to Y$

and sets A_1, A_2, B_1, B_2 s.t. $A_i \cap B_i \neq \emptyset$ &

$\mathrm{int}(A_1) \cup \mathrm{int}(B_1) = \Sigma X$ & $\Sigma f(A_1) \subseteq A_2$

$\mathrm{int}(A_2) \cup \mathrm{int}(B_2) = \Sigma Y$ & $\Sigma f(B_1) \subseteq B_2$

we get

$$\begin{tikzcd}
\cdots \arrow[r] & \tilde{H}_{n+1}(\Sigma X) \arrow[r] \arrow[d, "\Sigma f_*"] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{H}_n(A_1 \cap B_1) \arrow[r] \arrow[d, "f_*"'] & \tilde{H}_n(A_1) \oplus \tilde{H}_n(B_1) \arrow[r] \arrow[d] & \cdots \\
 & \tilde{H}_{n+1}(\Sigma Y) \arrow[r] & \tilde{H}_n(A_2 \cap B_2) \arrow[r] & \tilde{H}_n(A_2) \oplus \tilde{H}_n(B_2) \arrow[r] & \cdots
\end{tikzcd}$$

$\square$

A4 a) (i) A, B and $A \cap B$ are acyclic and in particular non-empty. Hence we can apply reduced Mayer Vietoris:

$$\dots \to \underbrace{\tilde{H}_n(A)}_{=0} \oplus \underbrace{\tilde{H}_n(B)}_{=0} \to \tilde{H}_n(A \cup B) \to \underbrace{\tilde{H}_{n-1}(A \cap B)}_{=0} \to \dots \quad \forall n$$

$$\Rightarrow \tilde{H}_n(A \cup B) = 0 \quad \forall n \quad \square$$

(ii) $\underline{n \geq 2}$: Mayer Vietoris gives

$$\dots \underbrace{H_n(A)}_{=0} \oplus \underbrace{H_n(B)}_{=0} \to H_n(A \cup B) \to \underbrace{H_{n-1}(A \cap B)}_{\underbrace{\geq 1}_{=0}}$$

$$\Rightarrow H_n(A \cup B) = 0 \quad \forall n \geq 2$$

$\underline{\text{for } n=1}$: Mayer Vietoris gives

$$\to \underbrace{H_1(A)}_{=0} \oplus \underbrace{H_1(B)}_{=0} \to H_1(A \cup B) \to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to$$

- if A and/or B empty $\Rightarrow A \cap B$ empty $\Rightarrow H_0(A \cap B) = 0$
  $\Rightarrow H_1(A \cup B) = 0$
- if A, B acyclic & $A \cap B$ empty $\Rightarrow H_0(A \cap B) = 0$
  $\Rightarrow H_1(A \cup B) = 0$
- if A, B & $A \cap B$ acyclic $\to$ see a)(i) $\square$

b) Mayer Vietoris applied to

$T = (A\cup B)\cup C$

$H_n(A\cup B) = 0 \quad \forall n \geq 1$ by a)(ii)

$(A\cup B)\cap C = (A\cap C)\cup(B\cap C)$

but $A\cap C$ and $B\cap C$ are both either empty or acyclic,

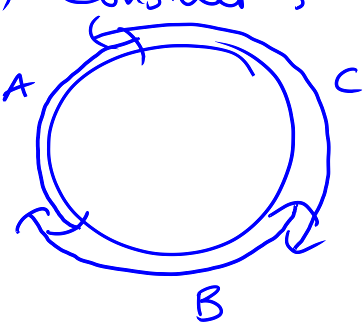
hence $H_n((A\cup B)\cap C) = 0 \quad \forall n\geq 1$ by a)(ii)

Thus

$$\cdots \to \underset{\substack{\| \\ 0}}{H_n(A\cup B)} \oplus \underset{\substack{\| \\ 0}}{H_n(C)} \to H_n((A\cup B)\cup C) \to \underset{\substack{\| \\ 0}}{H_{\underset{\geq 1}{n-1}}((A\cup B)\cap C)} \to \cdots$$

$\Rightarrow H_n(T) = H_n((A\cup B)\cup C) = 0 \quad \forall n \geq 2.$ $\square$

c) Consider S^1 and the subsets



$H_1(S^1) = H_0(S^1) = \mathbb{Z} \neq 0.$ $\square$

B5 a) $\begin{pmatrix} d_A & \mathrm{Id} & 0 \\ 0 & -d_A & 0 \\ 0 & -f & d_B \end{pmatrix} \circ \begin{pmatrix} d_A & \mathrm{Id} & 0 \\ 0 & -d_A & 0 \\ 0 & -f & d_B \end{pmatrix}$

$$= \begin{pmatrix} d_A \circ d_A & -d_A + d_A & 0 \\ 0 & d_A \circ d_A & 0 \\ 0 & f \circ d_A - d_B \circ f & d_B \circ d_B \end{pmatrix} = 0$$

b.c. f is a chain map, i.e. $f \circ d_A = d_A \circ f$ $\square$

b) $d_Z \circ \xi(b) = \begin{pmatrix} d_A & \mathrm{Id} & 0 \\ 0 & -d_A & 0 \\ 0 & -f & d_B \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ d_B(b) \end{pmatrix}$

$= \xi \circ d_B(b)$

$\eta \circ d_Z(a', a'', b) = \eta\big(d_A(a') + a'', -d_A(a''), -f(a'') + d_B(b)\big)$

$= f(d_A(a') + a'') - f(a'') + d_B(b)$

$= d_B \circ f(a') + d_B(b)$

b) (continued)

$$d_B \circ \eta\,(a', a'', b) = d_B(f(a') + b)$$
$$= d_B \circ f(a') + d_B(b)$$

c) Clearly $\eta \circ \xi(b) = b \Rightarrow \eta \circ \xi = \mathrm{Id}$

$$\xi \circ \eta(a', a'', b) = \xi(f(a') + b)$$
$$= (0, 0, f(a') + b)$$

Hence $$\xi \circ \eta - \mathrm{Id}_Z = \begin{pmatrix} -\mathrm{Id} & 0 & 0 \\ 0 & -\mathrm{Id} & 0 \\ f & 0 & 0 \end{pmatrix}$$

We need to find $s : Z_i \longrightarrow Z_{i+1}$

$$s \circ d_Z + d_Z \circ s = \xi \circ \eta - \mathrm{Id}_Z$$

Let $s(a', a'', b) := (0, -a', 0)$

Then

$$s = \begin{pmatrix} 0 & 0 & 0 \\ -\mathrm{Id} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $s \circ d_Z + d_Z \circ s =$

$$\begin{pmatrix} 0 & 0 & 0 \\ -\mathrm{Id} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} d_A & \mathrm{Id} & 0 \\ 0 & -d_A & 0 \\ 0 & -f & d_B \end{pmatrix}$$

$$+ \begin{pmatrix} d_A & \mathrm{Id} & 0 \\ 0 & -d_A & 0 \\ 0 & -f & d_B \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ -\mathrm{Id} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ -d_A & \mathrm{Id} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -\mathrm{Id} & 0 & 0 \\ d_A & 0 & 0 \\ f & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\mathrm{Id} & 0 & 0 \\ 0 & -\mathrm{Id} & 0 \\ f & 0 & 0 \end{pmatrix}$$

$$= \xi \circ \psi - \mathrm{Id}_Z \qquad \square$$

136 a) $A = X \times (1-\varepsilon, 1] \cup X \times [0, \varepsilon) / \sim$

$B = X \times \{1\} \cup X \times \{0\} / \sim = [X \times \{1\}]$

$\overline{B} \subseteq \operatorname{int}(A)$

$A \setminus B = X \times (1-\varepsilon) \cup X \times (0, \varepsilon)$

$T_f \setminus B = X \times (0,1)$

Then

$$H_*(X \times [0,1], X \times \partial I)$$
$$\cong H_*(X \times (0,1), X \times (0,\varepsilon) \cup X \times (1-\varepsilon, 1))$$
$$= H_*(T_f \setminus B, A \setminus B)$$
$$\cong H_*(T_f, A) \quad \text{by excision}$$
$$\cong H_*(T_f, X) \quad \text{as } A \text{ def retr. onto } X$$

b) $i_* : H_n(X \times \partial I) \longrightarrow H_n(X \times I)$

is surjective $\forall n$ as $X \times I$ is homotopy equivalent to $X \times \{0\}$ and to $X \times \{1\}$ and

$X \times \partial I = X \times \{0\} \cup X \times \{1\}$.

b) (continued)

Hence $H_n(X\times I) = \operatorname{im} i_* = \ker j_*$

$\Rightarrow j_* = 0 \;\forall n$

Then $H_{n+1}(X\times I, X\times\partial I) \cong \operatorname{Im}\partial_*$

$= \ker inc_*$

$= \{(\alpha,-\alpha) \mid \alpha\in H_n(X)\}$

$\cong H_n(X)$

c) Consider the l.e.s

$$\begin{tikzcd}
{} \arrow[r] & H_{n+1}(T_f, X_0) \arrow[r, "\partial_*"] \arrow[d, "g_*^{-1}"', "\cong"] & H_n(X_0) \arrow[r, "inc_*"] & H_n(T_f) \arrow[r] & \cdots \\
& H_{n+1}(X\times I, X\times\partial I) \arrow[d, "\partial_*"', "\cong"] & & & \\
& \operatorname{Im}(\partial_*) \arrow[d, equal] & & & \\
& \operatorname{Ker}(inc_*) \arrow[d, equal] & & & \\
& \{(\alpha,-\alpha) \mid \alpha\in H_n(X)\} \arrow[d, "\cong"] & & & \\
& H_n(X) \arrow[uuuuur, bend right, "\varphi"'] & & &
\end{tikzcd}$$

By the previous exercise we can replace $H_{n+1}(T_f, X)$ by $H_n(X)$ and the map $\varphi: H_n(X) \longrightarrow H_n(X_0)$ becomes

$$\varphi: \alpha \longmapsto (\alpha, -\alpha) \mapsto \partial_* \circ g_* \circ \partial_*^{-1}(\alpha, -\alpha).$$

By commutativity of the diagram $\partial_* \circ g_* \circ \partial_*^{-1}$ equals

$$g_*: H_n(X \times \partial I) \longrightarrow H_n(X_0)$$

Identifying $H_n(X_0) \cong H_n(X)$ we get the Wang sequence. $\square$