Algebraic Topology I

Problem set 1

Definition: For a map $f : X \to Y$, the *mapping cylinder* M_f is the quotient space of the disjoint union $Y \sqcup (X \times I)$ (where $I := [0,1]$) obtained by identifying each $(x,0) \in X \times I$ with $f(x) \in Y$.
The *mapping cone* C_f of f is defined as the quotient space $M_f/(X \times \{1\})$.
There are embeddings $i_X : X \to M_f;\ x \mapsto [x,1]$ resp. $i_Y : Y \to M_f;\ y \mapsto [y]$ of X resp. Y into the mapping cylinder of f.

Definition: $\mathbb{R}P^2$ is the topological space obtained from S^2 by identifying antipodal points, i.e. $\mathbb{R}P^2 = S^2/\sim$, where $x \sim y$ if and only if $x = \pm y$. Alternatively, we could define $\mathbb{R}P^2$ as the topological space obtained from the unit disk D^2 by identifying antipodal points on ∂D^2. I.e. $\mathbb{R}P^2 = D^2/\sim$, where $x \sim y$ if and only if $x, y \in \partial D^2$ and $x = \pm y$.

1. Let X be a contractible space and let $r : X \to A$ be a retraction. Show that A is also contractible.
2. Let $f, g : X \to S^n$ be maps such that $f(x) \neq -g(x)$ for all $x \in X$. Show that $f \simeq g$.
3. Let $f : X \to Y$ be a map. Suppose that there exist maps $g, h : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $h \circ f \simeq \mathrm{id}_X$. Show that f is a homotopy equivalence.
4. (a) Show that a map $\phi : M_f \to Z$ is continuous if and only if the induced maps $\phi_{X \times I} : X \times I \to Z$ and $\phi_Y : Y \to Z$ are continuous.

   (b) Let $f : X \to Y$ be a continuous map and let $r : M_f \to Y$ be the retract defined by $r([y]) = y$ and $r([x,t]) = f(x)$ for $y \in Y$, $x \in X$ and $t \in I$.
   Show that there is a commutative diagram

   $$\begin{array}{ccc} X & \xrightarrow{i_X} & M_f \\ & \searrow^{f} & \downarrow{\scriptstyle r} \\ & & Y \end{array}$$

   such that $id_{M_f} \simeq i_Y \circ r$ and hence $M_f \simeq Y$.

   (c) Show that two spaces X and Y have the same homotopy type if and only if they can be embedded as (weak) deformation retracts of the same space.

5. Consider the map $f : S^1 \to S^1$, $e^{2\pi it} \mapsto e^{4\pi it}$ (or $z \mapsto z^2$). Prove that $C_f \approx \mathbb{R}P^2$.