2. **Compute the simplicial homology groups of the Δ-complex obtained from $n+1$ 2-simplices $\Delta_0^2, \ldots, \Delta_n^2$ by identifying all three edges of Δ_0^2 to a single edge, and for $i > 0$ identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$ of Δ_i^2 to a single edge and the edge $[v_0, v_2]$ to the edge $[v_0, v_1]$ of Δ_{i-1}^2.**

Problem 6, §2.1 We begin by describing the equivalence classes of k-faces, $k = 0, 1, 2$. Let $\Delta_i[v_0^i v_1^i v_2^i]$.

- *The 0-faces.* We have

$$[v_0^0 v_1^0] \sim [v_1^0 v_2^0] \sim [v_0^0 v_2^0]$$

so that

$$v_0^0 \sim v_1^0 \sim v_2^0.$$

Denote by v^0 the equivalence class containing these vertices. Note that

$$[v_0^1 v_2^1] \sim [v_0^0 v_1^0] \Longrightarrow v_0^1 \sim v^0, \;\; v_2^1 \sim v^0$$
$$[v_0^1 v_1^1] \sim [v_1^1 v_2^1] \Longrightarrow v_1^1 \sim v^0.$$

Iterating this procedure we deduce that there exists a single equivalence class of vertices.

- *The 1-faces.* Denote by e_0 the equivalence class containing the edges of Δ_0. Then the edge $[v_0^1 v_2^1]$ belongs to this equivalence class. We also have another n-equivalence classes e_i containing the pair $[v_0^i v_1^i]$, $[v_1^i v_2^i]$. Observe that

$$[v_0^i v_2^i] \sim e_{i-1}, \;\; i = 1, \cdots, n.$$

- *The 2-faces.* We have $n+1$ equivalence classes of 2-faces, $\Delta_0, \Delta_1, \cdots, \Delta_n$.

- *$\partial : C_2 \to C_1$.* We have

$$C_2 = \mathbb{Z}\langle \Delta_0, \cdots, \Delta_1 \rangle, \quad C_1 = \mathbb{Z}\langle e_0, e_1, \cdots, e_n \rangle$$
$$\partial \Delta_0 = e^0, \;\; \partial \Delta_i = [v_0^i v_1^i] + [v_1^i v_2^i] - [v_0^i v_2^i] = 2e_i - e_{i-1}.$$

- *$\partial : C_1 \to C_0$.* We have

$$C_0 = \mathbb{Z}\langle v^0 \rangle$$

and

$$\partial e_i = 0, \;\; \forall i = 0, 1, \cdots, n.$$

- *Z_2 and H_2.* We have $B_2 = 0$ and

$$Z_2 = \Big\{ \sum_{i=0}^n x_i \Delta_i; \;\; \sum_{i=0}^n x_i \partial \Delta_i = 0 \Big\}$$

Thus

$$\sum_{i=0}^n x_i \Delta_i \in Z_2 \Longleftrightarrow \begin{cases} x_n & = & 0 \\ -x_n + 2x_{n-1} & = & 0 \\ \vdots & \vdots & \vdots \\ -x_2 + 2x_1 & = & 0 \\ -x_1 + x_0 & = & 0 \end{cases}$$

We deduce $Z_2 = 0$ so that $H_2 = 0$.

- *Z_1 and H_1.* We have $Z_1 = C_1$ and H_1 has the presentation

$$\Big\langle e_0, e_1, \cdots, e_n \big| \;\; 0 = 2e_n - e_{n-1} = \cdots 2e_1 - e_0 = e_0 \Big\rangle.$$

Hence

$$e_{n-1} = 2e_n, \;\; e_{n-2} = 2e_{n-1}, \cdots, e_0 = 2e_1 = 0$$

so that H_1 is the cyclic group of order 2^n generated by e_n. By general arguments we have $H_0 = \mathbb{Z}$.

$\square$

3. **Construct a 3-dimensional Δ complex X from n tetrahedra $T_1, \dots, T_n$ by the following two steps. First arrange the tetrahedra in a cyclic pattern as in the figure, so that each T_i shares a common vertical face with its two neighbors T_{i-1} and T_{i+1} , subscripts being taken mod n. Then identify the bottom face of T_i with the top face of T_{i+1} for each i. Show the simplicial homology groups of X in dimensions $0, 1, 2, 3$ are $\mathbb{Z}, \mathbb{Z}_n, 0, \mathbb{Z}$, respectively.**

Problem 8, §2.1 Hatcher. Denote by $[V_0^i V_1^i V_2^i V_3^i]$ the i-th 3-simplex.

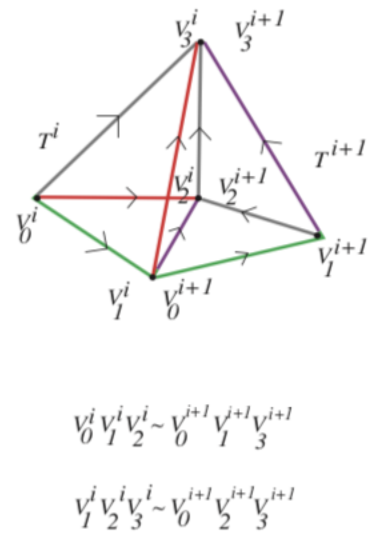

FIGURE 2. *Cyclic identifications of simplices*

To describe the associated chain complex we need to understand the equivalence classes of k-faces, $k = 0, 1, 2, 3$.

• <u>*0-faces.*</u> We deduce $V_0^i \sim V_0^{i+1}$ $\forall i$ mod n and we denote by U_0 the equivalence class containing V_0^i.

Similarly $V_1^i \sim V_1^{i+1}$ and we denote by U_1 the corresponding equivalence class. Since $V_1^i \sim V_0^{i+1}$ we deduce $U_0 = U_1$.

Now observe that $V_2^i \sim V_2^{i+1}$ and we denote by U_2 the corresponding equivalence class. Similarly the vertices V_3^i determine a homology class U_3 and we deduce from $V_2^i \sim V_3 i + 1$ that $U_2 = U_3$. Thus we have only two equivalence classes of vertices, U_0 and U_2. The vertices V_0^i, V_1^j belong to U_0 while the vertices V_2^i, V_3^j belong to U_2.

• <u>*1-faces.*</u> The simplex T^i has six 1-faces (edges) (see Figure 2).

A vertical edge $v_i = [V_2^i V_3^i]$.
A horizontal edge $h_i = [V_0^i V_1^i]$.
Two bottom edges: bottom-right $br_i = [V_1^i V_2^i]$ and bottom-left $bl_i = [V_0^i V_2^i]$.
Two top edges: top-right $tr_i = [V_1^i V_3^i]$ and top-left $tl_i = [V_0^i V_3^i]$.

Inspecting Figure 2 we deduce the following equivalence relations.

$$br_i \sim bl_{i+1}, \;\; tr_i \sim tl_{i+1}, \;\; v_i \sim v_{i+1}, \tag{0.1}$$

$$h_i \sim h_{i+1}, \;\; bl_i \sim tl_{i+1}, \;\; br_i \sim tr_{i+1}. \tag{0.2}$$

We denote by v the equivalence class containing the vertical edges and by h the equivalence class containing the horizontal edges.

Observe next that

$$bl_i \sim tl_{i+1} \sim tr_i, \;\; \forall i$$

so that $bl_i \sim tr_i$ for all i. Denote by e_i the equivalence class containing bl_i. Observe that

$$bl_i \sim tr_i \sim e_i,\;\; tl_i \sim e_{i-1},\;\; br_i \sim e_{i+1}.$$

We thus have $(n+2)$ equivalence classes of edges v, h and e_i, $i = 1, \cdots, n$.

• <u>*2-faces.*</u> Each simplex T^i has four 2-faces

A bottom face $B_i = [V_0^i V_1^i V_2^i]$.
A top face $\tau_i = [V_0^i V_1^i V_3^i]$.
A left face $L_i = [V_0^i V_2^i V_3^i]$.
A right face $R_i = [V_1^i V_2^i V_3^i]$.

We have the identifications

$$R_i \sim L_{i+1},\;\; B_i \sim \tau_{i+1}.$$

We denote by B_i the equivalence class of B_i, by L_i the equivalence class of L_i and by R_i the equivalence class of R_i. Observe that

$$R_i = L_{i+1},\;\; \forall i \mod n.$$

There are exactly $2n$ equivalence classes of 2-faces.

• <u>*3-faces.*</u> There are exactly n three dimensional simplices $T^1, \cdots, T^n$.

• <u>*The associated chain complex.*</u>

$$C_0 = \mathbb{Z}\langle U_0, U_2\rangle,\;\; C_1 = \mathbb{Z}\langle v, h, e_i;\;\; 1 \le i \le n\rangle$$

$$C_2 = \mathbb{Z}\langle B_i, R_j;\;\; 1 \le i, j, k \le n\rangle,\;\; C_3 = \mathbb{Z}\langle T^i;\;\; 1 \le i \le n\rangle.$$

The boundary operators are defined as follows.

• <u>$\partial : C_3 \to C_2$</u>

$$\partial T^i = R_i - L_i + \tau_i - B_i = R_i - R_{i-1} + B_{i-1} - B_i.$$

• <u>$\partial : C_2 \to C_1$</u>

$$\partial B_i = h + br_i - bl_i = h + e_{i+1} - e_i,\;\; \partial R_i = v - tr_i + br_i = v + e_{i+1} - e_i,$$

• <u>$\partial : C_1 \to C_0$</u>

$$\partial e_i = U_2 - U_0,\;\; \partial h = 0,\;\; \partial v = 0.$$

For every sequence of elements $x = (x_i)_{i\in\mathbb{Z}}$ we define its "derivative" to be the sequence

$$\Delta_i x = (x_{i+1} - x_i),\;\; i \in \mathbb{Z}.$$

Using this notation we can rewrite

$$\partial T^i = \Delta_{i-1} R - \Delta_{i-1} B,\;\; \partial B_i = h + \Delta_i e,\;\; \partial R_i = v + \Delta_i e.$$

• <u>*The groups of cycles.*</u>

$$Z_0 = C_0,$$

$$Z_1 = \Big\{ ah + bv + \sum_i k_i e_i \in C_1;\;\; a, b, k_i \in \mathbb{Z},\;\; \sum_i k_i = 0 \Big\}$$

$$= \mathrm{span}_{\mathbb{Z}} \Big\{ v, h, \Delta_i e;\;\; 1 \le i \le n \Big\}^1.$$

[1]Here we use the elementary fact that the subgroup of $\mathbb{Z}^n$ described by the condition $x_1 + \cdots + x_n = 0$ is a free Abelian group with basis $e_2 - e_1, e_3 - e_2, \cdots, e_n - e_{n-1}$, where (e_i) is the canonical basis of $\mathbb{Z}^n$

Suppose

$$c=\sum_i x_iB_i+\sum_j y_jR_j\in Z_2.$$

Then

$$0=\partial C=\Big(\sum_i x_i\Big)h+\big(\sum_j y_j\Big)v+\sum_i(x_i+y_i)\Delta_ie$$

(use Abel's trick[2])

$$=\Big(\sum_i x_i\Big)h+\big(\sum_j y_j\Big)v-\sum_i\Delta_i(x+y)e_{i+1}.$$

We deduce

$$\sum_i x_i=\sum_j y_j=0,\;\;\Delta_i(x+y)=\Delta_ix+\Delta_iy=0,\;\;\forall y.$$

The last condition implies that (x_i+y_i) is a constant α independent of i. Using the first two conditions we deduce

$$0=\sum_i(x_i+y_i)=n\alpha$$

so that $x_i=-y_i$, for all i. This shows

$$Z_2=\Big\{\sum_i x_i(B_i-R_i);\;\;x_i\in\mathbb{Z},\;\;\sum_i x_i=0\Big\}.$$

To find Z_3 we proceed similarly. Suppose

$$c=\sum_i x_iT^i\in Z_3.$$

Then

$$0=\partial c=\sum_i x_i\Delta_{i-1}(R-B)=-\sum_i(R_i-B_i)\Delta_ix=-\sum(\Delta_ix)R_i+\sum_i(\Delta_ix)B_i.$$

We deduce $\Delta_ix=0$ for all i, i.e. x_i is independent of i. We conclude that

$$Z_3=\Big\{xT;\;\;x\in\mathbb{Z};\;\;T=\sum_i T^i\Big\}$$

In particular we conclude $H_3\cong\mathbb{Z}$.

- *The groups of boundaries and the homology.* We have

$$B_0=\mathrm{span}_{\mathbb{Z}}(U_2-U_0)\subset\mathbb{Z}\langle U_0,U_2\rangle.$$

We deduce

$$H_0=Z_0/B_0=C_0/B_0=\mathbb{Z}\langle U_0,U_2\rangle/\mathrm{span}_{\mathbb{Z}}(U_2-U_0)\cong\mathbb{Z}.$$
$$B_1=\mathrm{span}_{\mathbb{Z}}(\partial B_i,\partial R_j;\;\;1\leq i,j\leq n)\subset\mathbb{Z}\langle h,v,e_i;\;\;1\leq i\leq n\rangle.$$

Thus H_1 admits the presentation

$$H_1=Z_1/B_1=\Big\langle h,v,\Delta_ie;\;\;h=v=-\Delta_ie,\sum_i\Delta_ie=0\;\;1\leq i\leq n\rangle$$

[2]Abel's trick is a discrete version of the integration-by-parts formula. More precisely if R is a commutative ring, M is an R-module, $(x_i)_{i\in\mathbb{Z}}$ is a sequence in R, $(y_i)_{i\in\mathbb{Z}}$ is a sequence in M then we have

$$\sum_{i=1}^n(\Delta_ix)\cdot y_i=x_{n+1}y_n-x_1y_0-\sum_{j=1}^n x_j\cdot(\Delta_{j-1}y).$$

Using the equality

$$\sum_{i=1}^{n} \Delta_i e = 0$$

we deduce $nh = nv = 0$. This shows $H_1 \cong \mathbb{Z}/n\mathbb{Z}$.

Using the fact that for every sequence $x_i \in \mathbb{Z}$ $i \in \mathbb{Z}/n\mathbb{Z}$ such that $\sum_i x_i = 0$ there exists a sequence $y_i \in \mathbb{Z}$, $i \in \mathbb{Z}/n\mathbb{Z}$ such that

$$x_i = \Delta_i y, \ \ \forall i.$$

Any element $c \in Z_2$ has the form

$$c = \sum_i x_i (R_i - B_i),$$

where $\sum_i x_i = 0$. Choose y_i as above such that $x_i = -\Delta_i y$, $\forall i \bmod n$. Then

$$c = \partial \sum_i y_i T^i$$

so that $Z_2 = B_2$, i.e. $H_2 = 0$.

□

5. Solutions (c), (d), (e) are included with Javaplex.