Solutions ex sheet 2

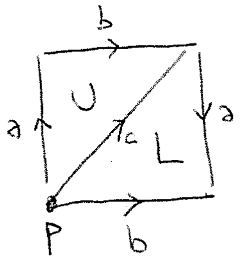
Ex 1

The torus and $\mathbb{RP}^2$ are in Hatcher: examples 2.3 and 2.4.

$$H_n^\Delta(T) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & n=1 \\ \mathbb{Z} & n=0,2 \\ 0 & \text{otherwise} \end{cases}$$

$$H_n^\Delta(\mathbb{RP}^2) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & n=1 \\ \mathbb{Z} & n=0 \\ 0 & \text{otherwise} \end{cases}$$

Klein bottle:

$K =$ 

$$\Delta_3(K)=0 \xrightarrow{\partial_3} \Delta_2(K)=\mathbb{Z}U\oplus\mathbb{Z}L \xrightarrow{\partial_2} \Delta_1(K)=\mathbb{Z}a\oplus\mathbb{Z}b\oplus\mathbb{Z}c \xrightarrow{\partial_1} \Delta_0(K)=\mathbb{Z}p \to$$

$$U \longmapsto a+b-c$$

$$L \longmapsto b-a-c$$

$$a,b,c \longmapsto 0$$

$$H_2^\Delta(K) = \ker(\partial_2) = \ker\left(\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}\right) = 0$$

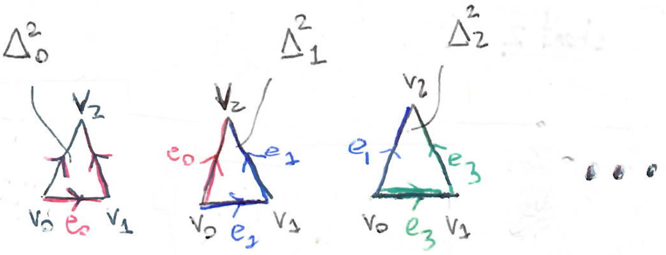
$$H_1^\Delta(K) = \frac{\ker(\partial_1)}{\mathrm{Im}(\partial_2)} = \frac{\Delta_1(K)}{\mathrm{Im}(\partial_2)} \cong \mathbb{Z}^2/_{0\oplus 2\mathbb{Z}} = \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$$

$$\cong \mathbb{Z}\begin{pmatrix}0\\2\\0\end{pmatrix}\oplus\mathbb{Z}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$c=-a-b$

$$H_0^\Delta(K) = \Delta_0(K) = \mathbb{Z}$$

Ex 2

$X =$ 

has an obvious Δ-structure explained in the picture.

$$0 \longrightarrow \Delta_2 X \xrightarrow{\partial_2} \Delta_1 X \xrightarrow{\partial_1} \Delta_0 X \xrightarrow{\partial_0} 0$$

$$\Delta^2_0 \longmapsto e_0$$

$$\Delta^2_i \longmapsto 2e_i - e_{i-1} \quad i=1,..,n$$

$$e_0 \longmapsto 0$$

$$e_i \longmapsto 0 \quad i>0$$

$$\Rightarrow \quad H_2^\Delta(X) = \ker(\partial_2) = \ker\left(\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 2 & & & \vdots \\ \vdots & 0 & \ddots & & 0 \\ \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & \cdots & & 2 \end{bmatrix}\right) = 0$$

$$H_1^\Delta(X) = \frac{\ker(\partial_1)}{\operatorname{Im}(\partial_2)} = \frac{\Delta_1 X \cong \mathbb{Z}^{n+1}}{\left\langle \begin{matrix} e_0 = 0 \\ 2e_1 = e_0 = 0 \\ 2e_2 = e_1 \\ 2e_3 = e_2 \\ \vdots \\ 2e_n = e_{n-1} \end{matrix} \right\rangle} \cong \mathbb{Z}/2^n\mathbb{Z}$$

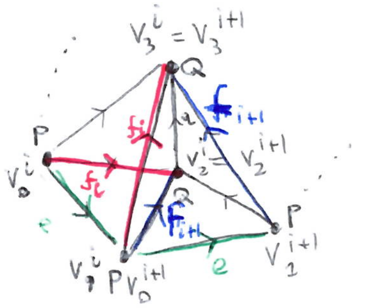
$\uparrow$

generator e_n $\left(2^n e_n = 2^{n-1} \cdot \underbrace{2 e_n}_{e_{n-1}} = \dots = 2e_1 = 0\right)$

$$H_0^\Delta X = \frac{\ker(\partial_0)}{\operatorname{Im}(\partial_1)} = \mathbb{Z}$$

Ex 3

Consider

$X =$ 

$\sigma_i := [v_0^i\, v_1^i\, v_2^i\, v_3^i]$ = ith simplex

$f_i := [v_0^i\, v_2^i]$

$e := [v_0^i\, v_1^i]$ (independent of i)

$a := [v_2^i\, v_3^i]$ (independent of i)

$R_i := [v_0^i\, v_2^i\, v_3^i] = L_i =: [v_1^{i-1}\, v_2^{i-1}\, v_3^{i-1}]$

$\tau_i := [v_0^i v_1^i v_3^i] = B_{i-1} := [v_0^{i-1} v_1^{i-1} v_2^{i-1}]$

where we are also glueing $[v_0^i\, v_1^i\, v_2^i] = [v_0^{i+1}\, v_1^{i+1}\, v_3^{i+1}]$

Give X the Δ-structure in the picture.

Consider the complex

$$\begin{array}{ccccccccc}
\Delta_3 X & \xrightarrow{\partial_3} & \Delta_2 X & \xrightarrow{\partial_2} & \Delta_1 X & \xrightarrow{\partial_1} & \Delta_0 X & \xrightarrow{\partial_0} & 0 \\
\| & & \| & & \| & & \| & & \\
\bigoplus_{i=1}^{n} \mathbb{Z}\sigma_i & \to & \bigoplus_{i=1}^{n} \mathbb{Z}\tau_i \oplus \bigoplus_{i=1}^{n} \mathbb{Z}L_i & \to & \mathbb{Z}a \oplus \mathbb{Z}e \oplus \bigoplus_{i=1}^{n} \mathbb{Z}f_i & \to & \mathbb{Z}P \oplus \mathbb{Z}Q & &
\end{array}$$

$$\sigma_i \longmapsto R_i - L_i + \tau_i - B_i$$

$$= L_{i-1} - L_i + \tau_i - \tau_{i+1}$$

$$\tau_i \longmapsto f_i - f_{i-1} + e$$

$$L_i \longmapsto f_i + a - f_{i-1}$$

$$a \longmapsto 0$$

$$e \longmapsto 0$$

$$f_i \longmapsto Q - P$$

$$P \longmapsto 0$$

$$Q \longmapsto 0$$

$$\Rightarrow H_3^{\Delta}(X) = \ker(\partial_3) = \ker\left(\overbrace{\begin{array}{c} \\ L_1 \\ \\ \\ L_n \\ \tau_1 \\ \tau_2 \\ \\ \\ \tau_m \end{array} \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & -1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & -1 \\ \hline 1 & 0 & 0 & \cdots & -1 \\ -1 & 1 & 0 & \cdots & \vdots \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & 0 & -1 & \ddots & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}}^{A} \right) = \mathbb{Z}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \mathbb{Z}$$

↳ This is a $(2n) \times n$ matrix

$$H_2^\Delta(X) = \frac{\ker(\partial_2)}{\mathrm{Im}(\partial_3)} = 0 \quad \text{because}$$

$$\ker(\partial_2) = \ker\left(\begin{array}{c|c} & \begin{array}{ccccccccc} \tau_1 & \tau_2 & \cdots & \tau_m & L_1 & & & & \end{array} \\ \begin{array}{c} f_1 \\ \\ \vdots \\ \\ f_m \\ a \\ e \end{array} & \left[\begin{array}{cccc|cccc} 1 & -1 & 0 \cdots & 0 & 1 & -1 & 0 \cdots & 0 \\ 0 & 1 & & \vdots & 0 & 1 & & \vdots \\ \vdots & 0 & \ddots & 0 & \vdots & 0 & \ddots & 0 \\ 0 & \vdots & & -1 & 0 & 0 & & -1 \\ -1 & 0 & \cdots 0 & 1 & -1 & 0 & \cdots 0 & 1 \\ \hline 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{array}\right] \end{array}\right) = \Big\{ \sum \underset{\mathbb{Z}}{\underset{\uparrow}{x_i}} (\tau_i - L_i) \,\Big|\, \sum x_i = 0 \Big\}$$

$$\overset{= \,\mathrm{Im}(\partial_3)}{\uparrow}$$

$$\cancel{\sum x_i(\tau_i - L_i)} = \partial\Big(\sum_i y_i \sigma_i\Big) = \sum_i y_i (L_{i-1} - L_i + \tau_i - \tau_{i+1}) =$$

$$= \sum_i (\tau_i - L_i)(y_i - y_{i+1})$$

and for $x = (x_1 \ldots x_n)$ with $x_1 + \ldots + x_n = 0$ the system

$$x_i = y_i - y_{i+1} \qquad \forall i = 1, \ldots, m \qquad (\text{here } i+1 = 1 \text{ when } i = m)$$

has ~~unique~~ solution exactly because as before

$$\ker(A) = \mathbb{Z}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \mathrm{Im}(A) = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \; x_1 + \ldots + x_n = 0 \right\}$$

$$H_1^\Delta(X) = \frac{\ker(\partial_1)}{\mathrm{Im}(\partial_2)} \overset{\uparrow}{=} \frac{\bigoplus_{i=1}^{m} \mathbb{Z}(f_i - f_{i+1}) \oplus \mathbb{Z}a \oplus \mathbb{Z}e}{\langle f_i - f_{i+1} = -e = -a \rangle}$$

$$\ker(\partial_1) = \langle f_i - f_j \rangle \oplus \mathbb{Z}a \oplus \mathbb{Z}e$$

$$H_0^\Delta(X) = \frac{\mathbb{Z}P \oplus \mathbb{Z}Q}{\langle P = Q \rangle} \cong \mathbb{Z}$$

Ex 4

$$f_* : H_0(X) \longrightarrow H_0(X)$$
$$[x] \longmapsto [f(x)]$$

so we only need to prove that $[f(x)] = [x]$ in $H_0(X)$.

More generally, $\left.\begin{array}{l} X \text{ path connected} \\ x, y \in X \end{array}\right\} \Rightarrow [x] = [y]$ in $H_0 X$

reason: let $\gamma : I \longrightarrow X$ be a path from x to y
then $\partial\gamma = y - x \Rightarrow [y-x] = [y] - [x] = 0$ in $H_0 X$.