Homework 3

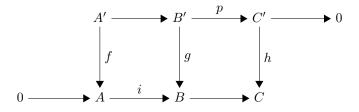
- 1. (a) Compute the fundamental group of $T \setminus \{p\}$, where T is the torus and p is any point in T.
 - (b) Compute the fundamental group of $K \setminus \{p\}$, where K is the Klein bottle and p is any point in K.
 - (c) Use Van Kampen's Theorem to compute $\pi_1(T)$ and $\pi_1(K)$ (Hint: Use 1 & 2).
 - (d) Use the Hurewicz Theorem to compute $H_1(T)$ and $H_1(K)$.
- 2. Let $f: (X, x_0) \to (Y, y_0)$ be a map of pointed spaces and consider the induced maps $f_{\#}: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $f_*: H_1(X) \to H_1(Y)$. Prove commutativity of the diagram

$$\begin{array}{c} \pi_1(X, x_0) \xrightarrow{f_{\#}} \pi_1(Y, y_0) \\ & \downarrow \phi_X \\ & \downarrow \phi_Y \\ H_1(X) \xrightarrow{f_*} H_1(Y) \end{array}$$

where ϕ_X and ϕ_Y are the Hurewicz homomorphisms.

- 3. Verify that $f \sim g$ implies $f_* = g_*$ for induced homomorphisms of reduced homology groups.
- 4. Prove the snake lemma.

Consider the following commutative diagram:



where the rows are exact. Then there is an exact sequence relating the kernels and cokernels of f, g, and h, where ∂ is a homomorphism ∂ : ker $h \to \operatorname{coker} f$, known as the connecting homomorphism.

$$\ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\partial} \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h.$$

5. For an exact sequence $A \to B \to C \to D \to E$ show that C = 0 iff the map $A \to B$ is surjective and $D \to E$ is injective. Hence for a pair of spaces (X, A), the inclusion $A \hookrightarrow X$ induces isomorphisms on all homology groups iff $H_n(X, A) = 0$ for all n.