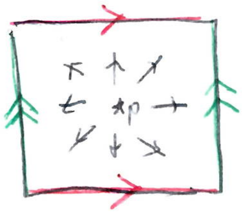
Homework 3 solutions

Ex 1

(a) Compute $\pi_1(T \setminus \{p\})$:

Sol

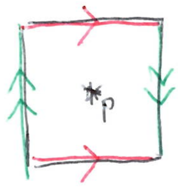
Write $T\setminus\{p\} =$  $\underset{\text{hom equivalent}}{\simeq}$ 

$\Rightarrow \pi_1(T\setminus\{p\}) \cong \mathbb{Z} * \mathbb{Z}$

$\blacksquare$

(b) compute $\pi_1(K\setminus\{p\})$.

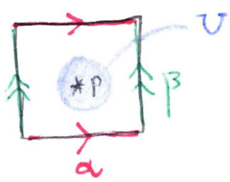
Sol

Write $K\setminus\{p\} =$  $\underset{\text{hom eq}}{\simeq}$  $\Rightarrow \pi_1(K-\{p\}) \cong \mathbb{Z} * \mathbb{Z}$

$\blacksquare$

(c) Use Van Kampen to compute $\pi_1(T)$ and $\pi_1(K)$

Sol

- T : let $U \underset{i}{\subseteq} T$ be the open set  ~~Inner~~ and $V \underset{j}{\subseteq} T$, $V = T\setminus\{p\}$

~~Van Kampen~~ $\cancel{\Rightarrow \pi_1(T) \cong}$ Since $\pi_1(U) = 0$

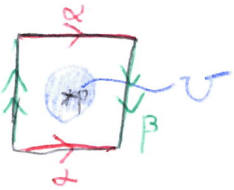
$$\underset{\text{Van Kampen}}{\Rightarrow} \pi_1(T) = \pi_1(V) \Big/ \underbrace{N\left(j_*(\pi_1(V\cap U))\right)}_{\text{smallest normal subgroup of } \pi_1(V) \text{ containing } j_*(\pi_1(V\cap U))}$$

Now $j_*\left(\pi_1(U \cap V)\right) = \langle \alpha\beta\alpha^{-1}\beta^{-1} \rangle$ so

$$\pi_1(T) = \mathbb{Z}\alpha * \mathbb{Z}\beta \Big/ \langle \alpha\beta\alpha^{-1}\beta^{-1} \rangle = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$$

• K: Similarly let

$U \underset{i}{\subseteq} K$ defined by



$K \setminus \{p\} =: V \underset{j}{\hookrightarrow} K$

Then as before we obtain

$$\pi_1(K) = \mathbb{Z}\alpha * \mathbb{Z}\beta \Big/ \langle \alpha\beta^{-1}\alpha^{-1}\beta^{-1} \rangle$$

∎

ⓓ Use Hurwicz Thm to compute $H_1(T)$ and $H_1(K)$

Sol

$$H_1(T) \underset{\text{Hurwicz}}{=} \pi_1(T)^{ab} = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$$

$$H_1(K) \underset{\text{Hurwicz}}{=} \pi_1(K)^{ab} = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \Big/ \langle -2\beta \rangle = \mathbb{Z}\alpha \oplus \mathbb{Z}/_{(2)}\,\beta \cong \mathbb{Z} \oplus \mathbb{Z}/_{(2)}$$

as you already know from ex1 of Homework 2.

∎

Ex2

Let $f:(X,x_0)\to(Y,y_0)$. We want to prove that the diagram

$$\begin{tikzcd}
\pi_1(X,x_0) \arrow[r, "f_\#"] \arrow[d, "\phi_X"'] & \pi_1(Y,y_0) \arrow[d, "\phi_Y"] \\
H_1(X) \arrow[r, "f_*"'] & H_1(Y)
\end{tikzcd}$$

(ϕ_X, ϕ_Y are the Hurwicz homomorphisms).

commute:

proof

For $\{\alpha\}\in\pi_1(X,x_0)$ we have

$$f_*\,\phi_X\{\alpha\} = f_*[\alpha] = [f\circ\alpha] \in H_1(Y)$$

and

$$\phi_Y\, f_\#\{\alpha\} = \phi_Y\{f\circ\alpha\} = [f\circ\alpha]\in H_1(Y)$$

∎

Ex 3 Prove that $f\sim g \Rightarrow f_* = g_* : \tilde H(X)\to\tilde H(Y)$

proof

Consider the ~~from~~ chain equivalence $\tilde P: C_nX\to C_{n+1}Y$

$$\begin{tikzcd}
 & & & & C_{-1}(X) \arrow[d, equal] & \\
\cdots \arrow[r] & C_2(X) \arrow[r, "\partial_X"] \arrow[d] \arrow[dl, "\tilde P"] & C_1(X) \arrow[r, "\partial_X"] \arrow[d, "g_*-f_*"] \arrow[dl, "\tilde P"] & C_0(X) \arrow[r, "\deg"] \arrow[d, "g_*-f_*"] \arrow[dl, "\tilde P"] & \mathbb{Z} \arrow[r] \arrow[d, equal] \arrow[dl, "\tilde P"] & 0 \\
\cdots \arrow[r] & C_2(Y) \arrow[r] & C_1(Y) \arrow[r, "\partial_Y"'] & C_0(Y) \arrow[r] & \mathbb{Z} \arrow[r] & 0 \\
 & & & & C_{-1}(Y) \arrow[u, equal] &
\end{tikzcd}$$

defined by

$$\tilde P\big|_{C_iX} = P\big|_{C_iX} \quad \text{for } i\geq 0$$, where $P=$ prism operator defined in class

$$0=\tilde P: C_{-1}X=\mathbb{Z}\to C_0(Y)$$

Then $\partial\tilde P+\tilde P\partial = g_*-f_*$ simply because $C_{\geq 0}(X)\xrightarrow{P} C_{\geq 0+1}(Y)$ was a chain equivalence.

∎

Ex 4

Prove the snake lemma

Sol

Consider the following commutative diagram

$$\begin{tikzcd}
 & A' \arrow[r] \arrow[d, "f"] & B' \arrow[r, "p"] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r, "i"'] & B \arrow[r] & C &
\end{tikzcd}$$

where rows are exact. Then since it is commutative we get a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \ker(f) \arrow[r, dashed, "j_\star"] \arrow[d] & \ker(g) \arrow[r, dashed, "p_\star"] \arrow[d] & \ker(h) \arrow[d] \arrow[ddll, rounded corners, "\partial"] & \\
 & A' \arrow[r, "j"'] \arrow[d, "f"] & B' \arrow[r, "p"'] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r, "i"'] \arrow[d] & B \arrow[r, "q"'] \arrow[d] & C \arrow[d] & \\
 & \mathrm{Coker}(f) \arrow[r, dashed, "i_\star"'] \arrow[d] & \mathrm{Coker}(g) \arrow[r, dashed, "q_\star"'] \arrow[d] & \mathrm{Coker}(h) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $\partial : \ker(h) \longrightarrow \mathrm{Coker}(f)$ is defined as follows:

let $c' \in \ker(h) \subset C' \Rightarrow \exists\, b' \in B' : p(b') = c'$ and we have $qg(b') = h(c') = 0$

$$\Rightarrow \exists!\, a \in A : i(a) = g(b')$$

One defines

$$\partial(c') := [a] \in \mathrm{Coker}(f)$$

Exactly as in the proof of the 'existence of the long exact sequence in homology' seen in class, one verifies that ∂ is a well-defined homo of groups and that

$$\ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\partial} \mathrm{coker}(f) \to \mathrm{coker}(g) \to \mathrm{Coker}(h) \qquad (\dagger)$$

is exact. We now redo these checks in the next pages.

Claim 1 : || ∂ is well-defined

proof of claim 1

if $p(b') = p(\tilde{b}') = c'$ $\quad b', \tilde{b}' \in B'$

then $\quad p(b' - \tilde{b}') = 0 \Rightarrow \exists\, a' \in A'$ s.t. $j(a') = b' - \tilde{b}'$

Write $g(b') = i(a)$ $\quad g(\tilde{b}') = i(\tilde{a})$ $\quad a, \tilde{a} \in A$. We want to show that

$[a] = [\tilde{a}]$ in $\mathrm{Coker}(f)$ $\Leftrightarrow$ $a - \tilde{a} \in \mathrm{Im}(f)$

But $\quad i(a - \tilde{a}) = g(b') - g(\tilde{b}')$ and

and $i(f(a')) = g\, j(a') = g(b' - \tilde{b}') = g(b') - g(\tilde{b}')$

and i injective

$\Rightarrow \quad a - \tilde{a} = f(a')$.

■

Claim 2 || ∂ is homo of groups

proof of claim 2

Let $c', \tilde{c}' \in C'$. Write $p(b') = c'$ $\quad p(\tilde{b}') = \tilde{c}$ $\quad b', \tilde{b}' \in B'$

Then $\quad c' + \tilde{c}' = p(b' + \tilde{b}')$ and we want:

$\partial(c' + \tilde{c}') \stackrel{?}{=} \partial c' + \partial \tilde{c}'$

It is enough to observe that:

for $a, \tilde{a} \in A$ s.t $i(a) = g(b')$ $\quad i(\tilde{a}) = g(\tilde{b}')$ then

$i(a + \tilde{a}) = i(a) + i(\tilde{a}) = \cancel{g(b' + \tilde{b}')} = g(b') + g(\tilde{b}') = g(b' + \tilde{b}')$

$\Rightarrow \quad \partial c' = [a] \quad \partial \tilde{c}' = [\tilde{a}] \quad \partial(c' + \tilde{c}') = [a + \tilde{a}] = [a] + [\tilde{a}] = \partial c' + \partial \tilde{c}'$ ■

Claim 3 || The sequence $(\dagger)$ is exact

proof

- $j_\star$ is injective because j is injective;
- $p_\star \circ j_\star = 0$ because $p \circ j = 0$;
- $\ker(p_\star) \subset \mathrm{Im}(j_\star)$: let $b' \in \ker(g)$ s.t $p_\star b' = 0$

$\Rightarrow \exists\, a' \in A : j(a') = b'$ and

$i\, f(a') = g\, j(a') = g(b') = 0$

so since i is injective $f(a') = 0$ i.e. $a' \in \ker(f$

- $\partial \circ p_* = 0$: this is because for $b' \in \ker(g)$

$$\partial p_*(b') = [\text{unique element } a \text{ in } A \text{ s.t. } i(a) = g(b') \underset{\substack{| \\ b' \in \ker(g)}}{=} 0]$$

$$= [0]$$

- $\ker(\partial) \subset \mathrm{Im}(p_*)$: let $c' \in \ker(\partial)$. This means that $\exists\, a' \in A'$ s.t. $if(a') = g(b')$ for some $b' \in B'$ s.t. $p(b') = c'$.
  Now we also have $p(b' - j(a')) = p(b') = c'$
  and $g(b' - j(a')) = g(b') - if(a') = 0$
  i.e. $b' - j(a') \in \ker(g)$ and $p_*(b' - j(a')) = c'$.

- $i_* \circ \partial = 0$: because for $c' \in \ker(h)$, $c' = p(b')$ $b' \in B'$ and $a \in A$ s.t. ~~we have~~ $i(a) = g(b')$ we have
  $\partial c' = [a]$ and $i_*[a] = [i(a)] = [g(b')] = 0$ in $\mathrm{Coker}(g)$

- $\ker(i_*) \subset \mathrm{Im}(\partial)$: if $i_*[a] = 0$ $a \in A$ then $i(a) = g(b')$ for some $b' \in B$ and so $\partial(g(b')) = [a]$.

- $q_* \circ i_* = 0$: because $q \circ i = 0$

- $\ker(q_*) \subseteq \mathrm{Im}(i_*)$: if $q_*[b] = 0$ for $[b] \in \mathrm{Coker}(g)$
  then $q(b) = h(c')$ for some $c' \in C'$
  Write $c' = p(b')$ for some $b' \in B'$
  Then $q(b - g(b')) = q(b) - \underbrace{qg}_{\substack{\| \\ hp}}(b') = h(c') - h(c') = 0$

  $\Rightarrow \exists\, a \in A$ s.t. $i(a) = b - g(b')$

  $\Rightarrow i_*[a] = [b]$

■

Ex 5

Consider an exact sequence

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\delta} E.$$

Show $\quad [C=0] \iff \left[\begin{array}{c}\delta \text{ injective} \\ \text{and } \alpha \text{ surjective}\end{array}\right]$

proof

($\Rightarrow$): $C=0 \Rightarrow A \xrightarrow{\alpha} B \to 0$ exact $\iff \alpha$ surjective

and $0 \to D \xrightarrow{\delta} E$ exact $\iff \delta$ injective

($\Leftarrow$) ~~α injective $\Leftarrow$~~ Consider the exact sequence

$$\underset{\substack{\uparrow \\ \alpha \text{ surj}}}{0} = \mathrm{Coker}(\alpha) = B/\alpha(A) \xrightarrow{\bar{\beta}} C \xrightarrow{\bar{\gamma}} \mathrm{Ker}(\delta) \underset{\substack{\uparrow \\ \delta \text{ inj}}}{=} 0$$

$\Rightarrow \quad C=0$

$\blacksquare$