Homework 4

Ex 1

ⓐ Yes, there is an exact sequence of the required form. Namely

$$0 \to \mathbb{Z}_4 \xrightarrow{i} \mathbb{Z}_8 \oplus \mathbb{Z}_2 \xrightarrow{q} \mathbb{Z}_4 \to 0$$
$$1 \longmapsto (2,1)$$

Note that the map i is injective because $(2,1)$ has order 4 in $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ and the quotient is a group of order 4 where $\overline{(1,0)}$ has order 4, so $\mathbb{Z}_8 \oplus \mathbb{Z}_2 / i(\mathbb{Z}_4) \cong \mathbb{Z}_4$

ⓑ We claim that A fits in an exact sequence $0 \to \mathbb{Z}_{p^m} \to A \to \mathbb{Z}_{p^n} \to 0$ iff. $A \cong \mathbb{Z}_{p^{m+t}} \oplus \mathbb{Z}_{p^{n-t}}$ with $0 \le t \le m$.

proof

($\Rightarrow$) If you have $0 \to \mathbb{Z}_{p^m} \xrightarrow{i} A \xrightarrow{f} \mathbb{Z}_{p^n} \to 0 \Rightarrow |A| = p^{n+m}$ and so

$A \cong \mathbb{Z}_{p^{m_1}} \oplus \dots \oplus \mathbb{Z}_{p^{m_k}}$ with $m_1, \dots, m_k \ge 0$ $\quad m_1 + \dots + m_k = m+n$.

Also since $\mathbb{Z}_{p^m} \subset A \Rightarrow \exists$ element in A of order p^m

$\to \max(m_1, \dots, m_k) \ge m$.

Since $A \underset{f}{\twoheadrightarrow} \mathbb{Z}_{p^n} \Rightarrow \exists x \in A$ s.t $f(x) = 1$. Now $i(1)$ and $x \in A$ generate A as a group $\Rightarrow k \le 2$

reason: Consider the projection $A = \mathbb{Z}_{p^{m_1}} \times \dots \times \mathbb{Z}_{p^{m_k}} \overset{q}{\twoheadrightarrow} \underbrace{\mathbb{Z}_p \times \dots \times \mathbb{Z}_p}_{k \text{ of them}} = \mathbb{Z}_p^{\oplus k}$

Then $q(i(1))$ and $q(x)$ generate $\mathbb{Z}_p^{\oplus k}$ as $\mathbb{Z}_p$ vector space $\Rightarrow k \le 2$

$$\Rightarrow A = \underset{\substack{\uparrow \\ m_1 \ge m}}{\mathbb{Z}_{p^{m_1}}} \oplus \underset{\substack{\uparrow \\ \text{possibly } m_2 = 0}}{\mathbb{Z}_{p^{m_2}}} = \mathbb{Z}_{p^{m+t}} \oplus \mathbb{Z}_{p^{n-t}} \qquad 0 \le t \le m$$

($\Leftarrow$) If $A = \mathbb{Z}_{p^{m+t}} \oplus \mathbb{Z}_{p^{n-t}}$, then you have

$$0 \to \mathbb{Z}_{p^m} \xrightarrow{i} \mathbb{Z}_{p^{m+t}} \oplus \mathbb{Z}_{p^{n-t}} \to \mathbb{Z}_{p^n} \to 0$$
$$1 \longmapsto (p^t, 1)$$

and the quotient is a group of order p^n with $[(1,0)]$ of order p^n

$$\Rightarrow \mathbb{Z}_{p^{m+t}} \oplus \mathbb{Z}_{p^{n-t}} \big/ i(\mathbb{Z}_{p^m}) \cong \mathbb{Z}_{p^n}.$$

∎

(c) We claim that A fits in an exact sequence $0 \to \mathbb{Z} \xrightarrow{i} A \xrightarrow{f} \mathbb{Z}_n \to 0$ iff $A \cong \mathbb{Z} \oplus \mathbb{Z}_d$ with $d \mid n$

proof

$(\Rightarrow)$ Since $\underset{\mathbb{Z}}{\overset{\cong}{i(\mathbb{Z})}} \subset A \underset{\text{classification theorem of finitely generated abelian groups}}{\Rightarrow} A \cong \mathbb{Z}^l \oplus \mathbb{Z}_{n_1} \oplus \dots \oplus \mathbb{Z}_{n_k}$ $\quad n_1 \mid \dots \mid n_k$, $l \geq 1$

Let p be a prime $p \mid n_1$. So $p \mid n_i$ $\forall i = 1, .., k$. Consider $\mathbb{Z}^l \twoheadrightarrow \mathbb{Z}_p^{\oplus l}$ and $\forall i$ $\mathbb{Z}_{n_i} \twoheadrightarrow \mathbb{Z}_p$. This gives $A \twoheadrightarrow \mathbb{Z}_p^l \oplus \mathbb{Z}_p^k = \mathbb{Z}_p^{\oplus l+k}$. So $l+k \leq 2$

Again A is generated by $i(1)$ and any $x \in A$ s.t. $f(x) = 1 \in \mathbb{Z}_n$. Since $l \geq 1 \Rightarrow (l,k) \in \{(1,1), (1,0)\}$. Notice that $(l,k) = (2,0)$ is not possible because otherwise $A = \mathbb{Z}^2$ and tensoring our exact sequence with $- \otimes_{\mathbb{Z}} \mathbb{Q}$ we would have $\mathbb{Q} \xrightarrow{i} \mathbb{Q}^2 \to 0$ exact which is impossible for dimensional reasons.

$\Rightarrow A \cong \mathbb{Z} \oplus \mathbb{Z}_d$ (possibly $d=1$). We want now to show that $d \mid n$.

say $i : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}_d$, $1 \mapsto (a,b)$. Then $\mathbb{Z}_n \cong \dfrac{\mathbb{Z} \oplus \mathbb{Z}_d}{\langle (a,b) \rangle} \cong \dfrac{\mathbb{Z}x \oplus \mathbb{Z}y}{\langle ax+by, dy \rangle}$

$\underset{\text{Smith normal form theorem}}{\Rightarrow} n = \det \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = ad \Rightarrow d \mid n.$

($\Leftarrow$) Let $A = \mathbb{Z} \oplus \mathbb{Z}_d$ with $ad = n$. Then we have an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathbb{Z} \oplus \mathbb{Z}_d \xrightarrow{f} \mathbb{Z}_n \longrightarrow 0$$

$$\begin{aligned} 1 &\longmapsto (a, 1) \\ (x, y) &\longmapsto [x - ay] \end{aligned}$$

$\blacksquare$

Ex 2

$\boxed{\emptyset \neq A \subset X \ , \ \tilde{H}_n(A) = 0 \ \forall n. \text{ Then } H_n(X,A) = \tilde{H}_n(X) \ \forall n}$

proof

Consider the exact sequence of the pair (X, A):

$$\cdots \to \underset{\overset{\|}{0}}{\tilde{H}_n(A)} \to \tilde{H}_n(X) \to H_n(X, A) \to \underset{\overset{\|}{0}}{\tilde{H}_{n-1}(A)} \to \cdots$$

Therefore $H_n(X) \longrightarrow H_n(X, A)$ must be an isomorphism

■

Ex 3

ⓐ Show that $H_0(X,A) = 0$ iff A meet each path component of X

proof

From the exact sequence of (X,A):

$$\cdots \to H_0(A) \longrightarrow H_0(X) \to H_0(X,A) \to 0.$$

we have $H_0(X,A) = 0 \iff H_0(A) \to H_0(X)$ is surjective and this happens precisely when A meet each path component of X.

The reason is that

$$H_0(A) = \bigoplus_{\substack{C \subseteq A \\ \text{conn. comp}}} \mathbb{Z} \longrightarrow H_0(X) = \bigoplus_{\substack{D \subset X \\ \text{conn comp}}} \mathbb{Z}$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [p] & \longmapsto & [p] \end{array}$$

■

(5) Show that

$[H_1(X,A)=0]$ iff $\left[\begin{array}{l} H_1(A) \to H_1(X) \text{ is surjective} \\ \text{and each path component of } X \text{ contains at most one} \\ \text{path component of } A \end{array}\right]$

proof

From the exact sequence of (X,A):

$$\dots \to H_1(A) \to H_1(X) \to H_1(X,A) \to H_0(A) \to H_0(X) \to \dots$$

we get

$$[H_1(X,A)=0] \Leftrightarrow \left[\begin{array}{l} (i)\ H_1(A) \twoheadrightarrow H_1(X) \text{ is surjective} \\ (ii) \text{ and } H_0(A) \overset{i}{\hookrightarrow} H_0(X) \text{ is injective} \end{array}\right]$$

and (ii) happens exactly when each path component of X contains at most one path component of A again because i is the map $H_0(A) = \bigoplus_{\substack{C\subseteq A \\ \text{conn. component}}} \mathbb{Z} \longrightarrow \bigoplus_{\substack{D\subseteq X \\ \text{conn comp}}} \mathbb{Z}$

$$[P] \longmapsto i[P]=[P]$$

∎

Ex 4 Show that $H_1(\mathbb{R},\mathbb{Q})$ is free abelian and find a basis.

Proof

Consider the exact sequence of $(\mathbb{R},\mathbb{Q})$:

$$\to H_1(\mathbb{Q}) \to \underset{\substack{\| \\ 0}}{H_1(\mathbb{R})} \to H_1(\mathbb{R},\mathbb{Q}) \hookrightarrow H_0(\mathbb{Q}) \to H_0(\mathbb{R}) \to H_0(\mathbb{R},\mathbb{Q}) \to 0$$

$$\Rightarrow H_1(\mathbb{R},\mathbb{Q}) = \ker\Big(H_0(\mathbb{Q}) = \bigoplus_{p\in\mathbb{Q}} \mathbb{Z}[p] \longrightarrow H_0(\mathbb{R}) = \mathbb{Z} \Big) =$$

$$[p] \longmapsto 1$$

$$= \langle [p]-[q] \mid p,q\in\mathbb{Q} \rangle \underset{(*)}{\cong} \bigoplus_{\substack{p\in\mathbb{Q} \\ p\neq 0}} \mathbb{Z}([p]-[0]) \subset H_0(\mathbb{Q})$$

Therefore $\{ [p] - [0] \mid p \in \mathbb{Q}, p \neq 0 \}$ is a basis of $H_1(\mathbb{R}, \mathbb{Q})$

justification of ⊛ : ⊛ follows from the fact that you can always write

$$[p] - [q] = ([p] - [0]) - ([q] - [0])$$

and so the obvious map $\bigoplus_{\substack{p \in \mathbb{Q} \\ p \neq 0}} \mathbb{Z}([p] - [0]) \to \langle [p] - [q] \mid p, q \in \mathbb{Q} \rangle$

is an isomorphism

∎

Ex 5
Let $A \subset V \subset X$ where $\emptyset \neq A \subset X$ closed, $V \subset X$ open and $A \subset V$ is a strong deformation retract

(a) Commutativity of the diagram

$$\begin{tikzcd}
\tilde{H}_p(X,A) \arrow[r, "i"] \arrow[d, "q_*^3"] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{H}_p(X,V) \arrow[d, "q_*^2"] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{H}_p(X-A, V-A) \arrow[l, "\alpha"'] \arrow[d, "q_*^1"] \\
\tilde{H}_p(X/A, A/A) \arrow[r, "j"'] & \tilde{H}_p(X/A, V/A) & \tilde{H}_p(X/A - V/A, V/A - A/A) \arrow[l, "\beta"]
\end{tikzcd}$$

follows from naturality of cohomology groups and commutativity of the diagram

$$\begin{tikzcd}
(X,A) \arrow[r] \arrow[d, "q"] \arrow[dr, phantom, "\circlearrowleft"] & (X,V) \arrow[d, "q"] \arrow[dr, phantom, "\circlearrowleft"] & (X-A, V-A) \arrow[l] \arrow[d] \\
(X/A, A/A) \arrow[r] & (X/A, V/A) & (X/A - V/A, V/A - A/A) \arrow[l]
\end{tikzcd}$$

(b)
- i is an iso because $\tilde{H}_p(V,A)=0$ $\forall n$ being $V \supseteq A$ strong def retraction + long exact sequence of the ~~pair~~ triple (X,V,A);
- α is iso because of excision theorem;
- j is iso because $\tilde{H}_p(V/A, A/A)=0$ $\forall n$ being $V/A \supseteq A/A = \text{point}$ strong deformation retraction + long exact sequence of the ~~pair~~ triple $(X/A, V/A, A/A)$;
- β is iso because of excision theorem.

(c) $q_*^1 : \tilde{H}_p(X-A, V-A) \xrightarrow{\sim} \tilde{H}_p(X/A - V/A, V/A - A/A)$ is an iso because q is an isomorphism on $X-A \xrightarrow{q} X-A$;

(d) if (X,A) is a good pair by the commutativity of the square on the left and the fact that q_*^1, α, β are isomorphisms $\Rightarrow$ also $q_*^2 = \beta\, q_*^1 \alpha^{-1}$ is an isomorphism.
Similarly since the left square is commutative and i, j, q_*^2 are iso $\Rightarrow$ also $q_*^3 = j^{-1} q_*^2 i$ is an iso.

(e) Consider the long exact sequence of (X,A):

$$\dots \to \tilde{H}_p(A) \to \tilde{H}_p(X) \to \tilde{H}_p(X,A) \to \tilde{H}_{p-1}(A) \to \tilde{H}_{p-1}(X) \to \dots$$

$\|$ by (d)

$\tilde{H}_p(X/A, A/A) = \tilde{H}_p(X/A)$ use long exact sequence of $(X/A, A/A)$ and note that $\tilde{H}_p(A/A)=0$ $\forall n$.

$\forall t$

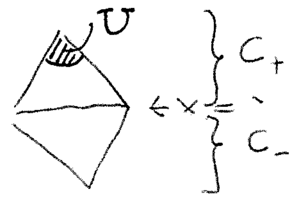
Ex 6

Let $\Sigma X :=$ quotient of $X \times [0,1]$ obtained by identifying $X \times \{0\}$ and $X \times \{1\}$ with points. Prove that there is a natural isomorphism $\tilde{H}_n(X) \xrightarrow{\sim} \tilde{H}_{n+1}(\Sigma X)$ $\forall n \geq 0$

proof

Let $C_+X := \{ [t,x] \mid t \geq \frac{1}{2} \} \subseteq \Sigma X$

picture ▸

$\Sigma X =$ 

$C_-X := \{ [t,x] \mid t \leq \frac{1}{2} \} \subseteq \Sigma X$

Identify $X = \{ [t,x] \mid t = \frac{1}{2} \} \subset \Sigma X$

Then Let U be a neighborhood of $p := [1,x] \in X$ s.t. $\overline{U} \subset C_+X \setminus X$

$\overline{U} \subset C_+X \setminus X$

By excission $\Rightarrow \tilde{H}_m(\Sigma X, C_+X) \cong \tilde{H}_m(\Sigma X - U, C_+X - U)$ (☆)

Since $(\Sigma X - U, C_+X - U)$ deformation retract to (C_-X, X):

$$\Rightarrow \quad \tilde{H}_m(\Sigma X - U, C_+X - U) \cong \tilde{H}_m(C_-X, X) \qquad (☆☆)$$

Consider now the exact sequence of the pair (C_-X, X):

$$①: \quad \underset{\substack{\uparrow \\ C_-X \text{ is contractible}}}{0 = \tilde{H}_n(C_-X)} \longrightarrow \tilde{H}_m(C_-X, X) \underset{\substack{\uparrow \\ \text{isomorphism}}}{\xrightarrow{\sim}} \tilde{H}_{m-1}(X) \longrightarrow \tilde{H}_{m-1}(C_-X) = 0$$

and the exact sequence of the pair $(\Sigma X, C_-X)$

$$②: \quad 0 = \tilde{H}_m(C_-X) \to \tilde{H}_m(\Sigma X) \underset{\substack{\uparrow \\ \text{iso}}}{\xrightarrow{\sim}} \tilde{H}_m(\Sigma X, C_-X) \to \tilde{H}_{m-1}(C_-X) = 0$$

$$\tilde{H}_m(\Sigma X, C_-X) \overset{\text{by ☆ and ☆☆}}{\cong} \tilde{H}_m(C_-X, X)$$

Then ① + ② $\Rightarrow$ $\tilde{H}_{m-1}(X) \cong \tilde{H}_m(C_-X, X) \cong \tilde{H}_m(\Sigma X)$ $\forall m$

Finally since each isomorphism is natural also the composition $\tilde{H}_{m-1}(X) \to \tilde{H}_n(\Sigma X)$ is natural.