Solutions to problem set 5

1. Denote by $D_1, D_2, D_3 \subset X$ the images of the three discs that get glued together. With $A := D_1 \cup D_2$ and $B := D_2 \cup D_3$, we have $X = A \cup B$. We will apply Mayer-Vietoris to (X, A, B). (The fact that the interiors of A and B do *not* cover X does not cause a problem here, because A and B have open neighbourhoods A' and B' which deformation retract onto them, obtained by adding a small "collar" neighbourhood to each, and such that $A' \cap B'$ deformation retracts onto $A \cap B$. Convince yourself of that!).

   A and B are homeomorphic to S^n, and $A \cap B = D_2$ is a copy of D^n. Hence $H_*(A \cap B)$ vanishes except in degree 0, and $H_*(A)$, $H_*(B)$ vanish except in degrees 0 and n. This implies that the third arrow in the following portion of the Mayer-Vietoris sequence is an isomorphism for all $k \geq 2$:

   $$0 \to H_k(A \cap B) \to H_k(A) \oplus H_k(B) \to H_k(X) \to H_{k-1}(A \cap B) \to \dots$$

   i.e., $H_k(X) \cong H_k(A) \oplus H_k(B)$ for all $k \geq 2$. For $k = 1$, we obtain

   $$0 \to H_1(A) \oplus H_1(B) \to H_1(X) \to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(X) \to 0$$

   Note that $H_0(A \cap B) \to H_0(A) \oplus H_0(B)$ is injective as $A \cap B = D_2$ is path-connected, and hence $H_1(X) \to H_0(A \cap B)$ is zero by exactness. Hence we obtain an isomorphism $H_1(X) \cong H_1(A) \oplus H_1(B)$ also in this case. Moreover, it is clear that $H_0(X) \cong \mathbb{Z}$ as X is path-connected. Putting all together, we obtain

   $$H_k(X) = \begin{cases} \mathbb{Z}, & k = 0, \\ \mathbb{Z}^2, & k = n, \\ 0, & \text{otherwise.} \end{cases}$$

2. We view $\mathbb{R}P^2$ as $D^2/\sim$, the quotient of D^2 obtained by identifying antipodal points on $S^1 = \partial D^2$. Let $A \subset \mathbb{R}P^2$ be the image of the interior of D^2 under the projection $D^2 \to \mathbb{R}P^2$, and let $B \subset \mathbb{R}P^2$ be the image of a collar neighbourhood of ∂D^2, homeomorphic to $[0, \varepsilon) \times \partial D^2$. The subsets form an open cover of $\mathbb{R}P^2$, and $A \cap B \simeq S^1$. The Mayer-Vietoris sequence for $(\mathbb{R}P^2, A, B)$ is

   $$\begin{aligned} 0 \to H_2(\mathbb{R}P^2) \to & H_1(A \cap B) \to H_1(A) \oplus H_1(B) \to H_1(\mathbb{R}P^2) \\ \to & H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(\mathbb{R}P^2) \to 0, \end{aligned}$$

   Note that $H_0(A \cap B) \to H_0(A) \oplus H_0(B)$ is injective since $A \cap B \simeq S^1$ is path-connected; moreover $H_1(A \cap B) = \mathbb{Z} = H_1(B)$, and $H_1(A) = 0$. Hence we obtain an exact sequence

   $$0 \to H_2(\mathbb{R}P^2) \to \mathbb{Z} \to \mathbb{Z} \to H_1(\mathbb{R}P^2) \to 0.$$

   The middle map $\mathbb{Z} \to \mathbb{Z}$ is multiplication by 2 because the canonical map $D^2 \to \mathbb{R}P^2$ restricts to a degree 2 map on ∂D^2. It follows that $H_2(\mathbb{R}P^2) = 0$ and $H_1(\mathbb{R}P^2) = \mathbb{Z}_2$; moreover $H_0(\mathbb{R}P^2) = \mathbb{Z}$ as $\mathbb{R}P^2$ is path-connected.

3. Let $A \subset K$ be the image of the interior of I^2 under the projection $I^2 \to K$, and let $B \subset K$ be the image of a neighbourhood of ∂I^2 homeomorphic to $[0, \varepsilon) \times \partial I^2$. Note that A is homeomorphic to D^2, while B deformation retracts onto the image of ∂I^2 in K, which is homeomorphic to $S^1 \vee S^1$. The intersection $A \cap B$ is homotopy equivalent to S^1. By the

same arguments as in the previous problem, the Mayer-Vietoris sequence for (K, A, B) yields an exact sequence of the form

$$0 \to H_2(K) \to H_1(A \cap B) \to H_1(B) \to H_1(K) \to 0$$

For suitable identitifications $H_1(A \cap B) \cong \mathbb{Z}$ and $H_1(B) \cong \mathbb{Z}^2$, the middle map is $\mathbb{Z} \xrightarrow{(0,2)} \mathbb{Z}^2$. It follows that $H_2(K) = 0$, $H_1(K) = \mathbb{Z}_2 \oplus \mathbb{Z}$; moreover, $H_0(K) = \mathbb{Z}$ as K is path-connected.

4. Denote by P our polygon and by $p : P \to \Sigma_g$ the canonical projection. Let A be the image of the interior of P under p, and let B be the image of a neighbourhood of ∂P which is homeomorphic to $[0, \varepsilon) \times S^1$. From the Mayer-Vietoris sequence for (Σ_g, A, B) we obtain an exact sequence
$$0 \to H_2(\Sigma_g) \to H_1(A \cap B) \to H_1(B) \to H_1(\Sigma_g) \to 0$$
by the same argument as in the previous problems. The image of ∂P under p, considered as a loop in Σ_g, represents the class $[a_1][b_1][a_1^{-1}][b_1^{-1}] \dots [a_g][b_g][a_g^{-1}][b_g^{-1}]$ in $\pi_1(B)$, which lies in the commutator subgroup of $\pi_1(B)$; hence its image in $H_1(B)$ vanishes. The map $H_1(A \cap B) \to H_1(B)$ is therefore zero, and we obtain $H_2(\Sigma_g) \cong H_1(A \cap B) \cong \mathbb{Z}$ and $H_1(\Sigma_g) \cong H_1(B) \cong \mathbb{Z}^{2g}$ by exactness and because $A \cap B \simeq S^1$ and $B \simeq S_1 \vee \cdots \vee S^1$ ($2g$ times). $H_0(\Sigma_g) \cong \mathbb{Z}$ is clear because Σ_g is path-connected.

5. We will show by induction that $H_0(\Sigma_g) \cong \mathbb{Z}$, $H_1(\Sigma_g) \cong \mathbb{Z}^{2g}$, and $H_2(\Sigma_g) \cong \mathbb{Z}$. For $\Sigma_1 = T^2$, this can be shown using e.g. cellular homology. To show it holds for Σ_{g+1} (assuming it's already shown for Σ_k, $k = 1, \dots, g$), consider the cover of Σ_{g+1} by open subsets A and B, where A and B are homeomorphic to $\Sigma_g^* := \Sigma_g \setminus \{\text{pt}\}$ resp. $\Sigma_1^* := \Sigma_1 \setminus \{\text{pt}\}$; the existence of such a cover is indicated by the definition of Σ_{g+1} as the connected sum of Σ_g and Σ_1.

   The punctured surface Σ_g^* is homotopy equivalent to $S^1 \vee \cdots \vee S^1$, a wedge of $2g$ circles (one can see this using the polygon description of Σ_g given in problem 5.5), and hence $H_2(\Sigma_g^*) \cong 0$, $H_1(\Sigma_g^*) \cong \mathbb{Z}^{2g}$ as one can see using e.g. cellular homology. Since moreover the intersection $A \cap B$ is homotopy equivalent to S^1, the Mayer-Vietoris sequence for (Σ_{g+1}, A, B) is

$$0 \to H_2(\Sigma_{g+1}) \to H_1(A \cap B) \to H_1(A) \oplus H_1(B) \to H_1(\Sigma_{g+1}) \to H_0(A \cap B) \to \dots \qquad (1)$$

   The homomorphism $H_1(A \cap B) \to H_1(A) \oplus H_1(B)$ is zero because $A \cap B$ deformation retracts onto a loop which lies in the commutator subgroup of both $\pi_1(A)$ and $\pi_1(B)$ (using the polygon description, see the solution of problem 5.5); the homomorphism $H_1(\Sigma_{g+1}) \to H_0(A \cap B)$ is zero because $A \cap B$ is path-connected, and hence the inclusions of S^1 into A and B induce injective maps on H_0. We therefore obtain isomorphisms $H_2(\Sigma_{g+1}) \cong H_1(S^1) \cong \mathbb{Z}$ and $H_1(\Sigma_{g+1}) \cong H_1(\Sigma_g^*) \oplus H_1(\Sigma_1^*) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}^2 = \mathbb{Z}^{2g+2}$, as required. $H_0(\Sigma_{g+1}) \cong \mathbb{Z}$ is clear as Σ_{g+1} is path-connected.

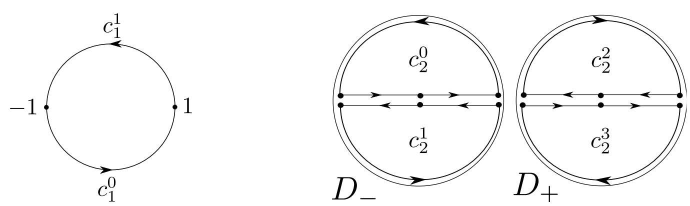
   (*Remark.* Using the polygon description of Σ_g is a bit of a short-cut, with which one could dispense as follows. First one computes inductively that $H_2(\Sigma_g^*) = 0$ and $H_1(\Sigma_g^*) = \mathbb{Z}^{2g}$, viewing Σ_{g+1} as the connected sum of Σ_g and Σ_1^* and applying Mayer-Vietoris to $(\Sigma_{g+1}^*, \Sigma_g^*, \Sigma_1^{**})$, where Σ_1^{**} denotes a twice-punctured torus. To get the induction started, one computes that $H_2(\Sigma_1^*) = 0 = H_2(\Sigma_1^{**})$, $H_1(\Sigma_1^*) = \mathbb{Z}^2$ and $H_1(\Sigma_1^{**}) = \mathbb{Z}^3$ using cellular homology. The Mayer-Vietoris sequence for $(\Sigma_{g+1}^*, \Sigma_g^*, \Sigma_1^{**})$ yields an exact sequence

$$0 \to H_2(\Sigma_{g+1}^*) \to H_1(S^1) \to H_1(\Sigma_g^*) \oplus H_1(\Sigma_1^{**}) \to H_1(\Sigma_{g+1}^*) \to 0.$$

   The map $H_1(S^1) \to H_1(\Sigma_g^*) \oplus H_1(\Sigma_1^{**})$ is injective because the intersection $\Sigma_g^* \cap \Sigma_1^{**} \simeq S^1$ generates a $\mathbb{Z}$-summand in $H_1(\Sigma_1^{**})$ (check this using cellular homology), and hence the second component of the map is non-zero. This implies that $H_2(\Sigma^*_{g+1}) = 0$ and $H_1(\Sigma^*_{g+1}) \cong \mathbb{Z}^{2g+2}$, using the inductive hypothesis. Having computed $H_*(\Sigma^*_g)$, one proceeds with the inductive computation of $H_*(\Sigma_{g+1})$ using the Mayer-Vietoris sequence (1); the vanishing of $H_1(A \cap B) \to H_1(A) \oplus H_1(B)$ can be checked by constructing inductively 2-chains in A and B whose boundary is a circle homotopy equivalent to $A \cap B$.)

6. For $S^0 = \{1, -1\}$, set $c_0 := 1 - (-1) \in \Delta_0(S^0)$. Then $[c_0] = [1] - [-1]$ generates $\widetilde{H}_0(S^0)$. (Recall that $\widetilde{H}_0(S^0)$ is the kernel of the canonical map $H_0(S^0) \to H_0(\mathrm{pt})$, which maps both generators $[-1]$ and $[1]$ of $H_0(S^0)$ to $[\mathrm{pt}]$.)

   As for S^1, consider 1-simplices $c_1^0, c_1^1 : \Delta_1 \to S^1$ as indicated in the figure below, and let $c_1 := c_1^0 + c_1^1 \in \Delta_1(S^1)$, which is a cycle. Consider now the cover of S^1 by $A = \operatorname{im} c_1^0$ and $B = \operatorname{im} c_1^1$ and the reduced Mayer-Vietoris sequence for (S^1, A, B) (which exists despite the fact that the interiors of A and B do not cover S^1, by the same reason as in problem 5.2). Then $A \cap B = S^0$, and the boundary morphism $\partial_* : \widetilde{H}_1(S^1) \to \widetilde{H}_0(A \cap B) = \widetilde{H}_0(S^0)$ takes $[c_1]$ to $[\partial c_1^0] = [1 - (-1)] = [c_0]$, our generator of $\widetilde{H}_0(S^0)$.



   To find a generator of $\widetilde{H}_2(S^2)$, view S^2 as the union of two copies D_- and D_+ of D^2, identified along their boundaries. Consider simplices $c_2^0, c_2^1, c_2^2, c_2^3 : \Delta_2 \to S^2$ as indicated in the figure; note that $c_2 = c_2^0 + c_2^1 + c_2^2 + c_2^3 \in \Delta_2(S^2)$ is a cycle. Consider now the Mayer-Vietoris sequence for (S^2, D_-, D_+). We have $D_- \cap D_+ = S^1$, and the boundary operator $\partial_* : H_2(S^2) \to H_1(D_- \cap D_+) = H_1(S^1)$ maps $[c_2]$ to $[\partial c_2^0 + \partial c_2^1] \in H_1(D_- \cap D_+)$, which is precisely $[c_1^0 + c_1^1] = [c_1]$.

7. Set $A = X \cup V$ and $B = Y \cup U$, thinking of these as subsets of $X \vee Y$. Then A deformation retracts onto X, B deformation retracts onto Y, and $A \cap B$ deformation retracts onto $\{*\}$, the point at which the two spaces are joint. The reduced Mayer-Vietoris sequence for $(X \vee Y, A, B)$ takes the form

$$\cdots \to \widetilde{H}_k(\{*\}) \to \widetilde{H}_k(X) \oplus \widetilde{H}_k(Y) \to \widetilde{H}_k(X \vee Y) \to \widetilde{H}_{k-1}(\{*\}) \to \dots,$$

   which yields $\widetilde{H}_k(X \vee Y) \cong \widetilde{H}_k(X) \oplus \widetilde{H}_k(Y)$ for all k, because $\widetilde{H}_k(\{*\}) = 0$ for all k.