Solutions to Homework 7

1. (a) The map

$$g_n\colon \mathbb{R}^{n+1}\backslash\{0\} \longrightarrow S^n$$
$$\underline{x} \longmapsto \frac{\underline{x}}{||\underline{x}||}.$$

descends to a homeomorphism $\mathbb{R}P^n \to S^n/(\underline{x} \sim -\underline{x})$. The map

$$f_n\colon B^n \longrightarrow \mathbb{R}P^n$$
$$x = (x_1, \dots, x_n) \longmapsto [x_1, \dots, x_n, \sqrt{1-|x|^2}]$$

descends to a homeomorphism $(B^n/\sim) \to \mathbb{R}P^n$, where $x \sim y$ in B^n if and only if $x = -y \in \partial B^n$.

(b) $\mathbb{R}P^0$ is a point and so it's a CW-complex with one 0-cell. View $\mathbb{R}P^n$ as $B^n/\sim$. As such, $\mathbb{R}P^n$ can be obtained as a 2-cell B^n glued to $\partial B^n/(x \sim -x)$ along the boundary via the projection $\partial B^n \to \partial B^n/(x \sim -x)$. Note that

$$\partial B^n/(x \sim -x) \approx S^{n-1}/(\underline{x} \sim -\underline{x}) \approx \mathbb{R}P^{n-1}.$$

Hence $\mathbb{R}P^n$ is obtained by gluing precisely one n-cell to $\mathbb{R}P^{n-1}$. This provides CW-structures as claimed by proceeding inductivly over

$$\mathbb{R}P^0 \subset \mathbb{R}P^0 \cup B^1 \approx \mathbb{R}P^1 \subset \mathbb{R}P^1 \cup B^2 \approx \mathbb{R}P^2 \subset \dots.$$

The characteristic map for the k-cell a_k is $f_{a_k} := f_k\colon B^k \to \mathbb{R}P^k \subset \mathbb{R}P^n$. Note that f_{a_k} is an embedding on $Int(B^k)$. Moreover, $f_{a_k}(\partial B^k) = \{[x_1, \dots, x_k, 0] \in \mathbb{R}P^k\} \approx \mathbb{R}P^{k-1} \subset \mathbb{R}P^n$. The attaching map is its restriction to ∂B^k:

$$f_{\partial a_k}\colon \partial B^k \approx S^{k-1} \longrightarrow \mathbb{R}P^{k-1} \subset \mathbb{R}P^n.$$

(c) The cellular chain complex of $\mathbb{R}P^n$ has one copy of $\mathbb{Z}$ in each degree $0 \le k \le n$ and is 0 in all the other degrees. For the k-cell a_k consider the projection

$$p_{a_k}\colon \mathbb{R}P^k \approx \left(B^k/\sim\right) \to \left(B^k/\partial B^k\right) \approx S^k.$$

The differential $d_k\colon \mathbb{Z} \longrightarrow \mathbb{Z}$ in degree $1 \le k \le n$ is given by multiplication with the degree of the map $p_{a_{k-1}} f_{\partial a_k} : S^{k-1} \to S^{k-1}$, $1 \le k \le n$. $[0] \in B^{k-1}/\partial B^{k-1} \approx S^{k-1}$ has two preimages under $p_{a_{k-1}} f_{\partial a_k}$: $N = (0, \dots, 0, 1) \in S^{n-1}$ and $S = (0, \dots, 0, -1) \in S^{n-1}$. Near N, this map is an orientation-preserving homeomorphism. So the local degree at N is 1. Near S, it is the antipodal map composed with an orientation-preserving homeomorphism. So the local degree near S is $(-1)^k$. Therefore,

$$\deg(p_{a_{k-1}} f_{\partial a_k}) = 1 + (-1)^k = \begin{cases} 0, & k \text{ odd}, \\ 2, & k \text{ even} \end{cases}$$

Suppose n is even. Then the cellular chain complex is

$$0 \to \mathbb{Z} \xrightarrow{*2} \mathbb{Z} \xrightarrow{0} \dots \mathbb{Z} \xrightarrow{*2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

with non-zero groups exactly in degrees $0, \ldots, n$, and thus we obtain

$$H_k(\mathbb{R}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z}/2\mathbb{Z}, & k = 1, 3, \ldots, n-1 \\ 0 & \text{otherwise.} \end{cases}$$

For n being odd, one computes similarly

$$H_k(\mathbb{R}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & k = 0, n \\ \mathbb{Z}/2\mathbb{Z}, & k = 1, 3 \ldots, n-2 \\ 0 & \text{otherwise.} \end{cases}$$

An alternative solution can be found in Bredon, Chapter IV. 14.

2. Compactify $\mathbb{R}^2$ and consider the stereographic projection

$$\pi \colon S^2 \to \mathbb{R}^2 \cup \{\infty\}.$$

View the graph G in S^2 by considering $\tilde{G} := \pi^{-1}(G) \subset S^2$. $\tilde{G}$ defines a CW-structure on S^2 with one 0-cell for each vertex of G, one 1-cell for each edge of G and one 2-cell for each face of G.

The Euler characteristic of S^2 therefore is $\xi(S^2) = v - e + f$. On the other hand, $\xi(S^2) = 2$, as can been seen from singular homology. We conclude: $v - e + f = 2$.

3. We view $T^3 = I^3/\sim$ as the quotient space of the cube I^3 under the relation that identifies opposite faces of the boundary. From this description, one sees that T^3 has a CW complex structure with one 0-cell a (any of the corner points—note that these get identified under $I^3 \to T^3$), three 1-cells b_1, b_2, b_3 (the line segments on the coordinate axes), three 2-cells c_1, c_2, c_3 (the squares in the coordinate planes), and one 3-cell d (all of I^3); in all these cases the attaching maps is given by restriction of the quotient map $I^3 \to T^3$.

The corresponding cellular chain complex is

$$0 \to \mathbb{Z} \xrightarrow{\partial_3} \mathbb{Z}^3 \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z} \to 0$$

with linear maps ∂_i which we now compute. We have $\partial_1 = 0$ since the attaching maps $f_{b_i} : I \to (T^3)^{(0)} = \{a\}$ take both boundary points $0, 1 \in I$ to the same point (cf. the remark in Bredon after Theorem 10.3). We also have $\partial_2 = 0$, since all maps $p_{b_i} f_{\partial c_j} : \partial I^2 \to S^1$ have degree 0 (by the same argument as for the standard CW complex structure of the 2-torus; see Bredon example 10.5).

As for ∂_3, consider any of the maps $p_{c_i} f_{\partial d} : \partial I^3 \to S^2$. Note that there are two opposite faces of ∂I^3 in whose interiors this map restricts to a homeomorphism, and that the map collapes the rest of ∂I^3 to a point in S^2. The degree of $p_{c_i} f_{\partial d}$ is hence the sum of the two local degrees at any two points q, q' in the two first-mentioned faces which get mapped to the same point in T^3. Now note that the restrictions of $p_{c_i} f_{\partial d}$ to these faces are obtained from one another by precomposing with an orientation-*reversing* map (for orientations induced from an orientation of ∂I^3); therefore the sum of these local degrees vanishes. It follows that also $\partial_3 = 0$.

Summing up, we obtain

$$H_i(T^3) \cong \begin{cases} \mathbb{Z}, & i = 0, 3, \\ \mathbb{Z}^3, & i = 1, 2. \end{cases}$$

4. (a) One possible CW complex structure has two 0-cells a_1, a_2 (the north and south poles), two 1-cells b_1, b_2 (the line segment mentioned in the description of X and another segment on the sphere connecting the poles), and one 2-cell c. We then have

$$\deg(p_{a_2} f_{\partial b_j}) = 1, \quad \deg(p_{a_1} f_{\partial b_j}) = -1$$

for $j = 1, 2$, supposing that the attaching maps $f_{b_j} : I \to X^{(0)}$ are such that both map $0 \in \partial I$ to a_1 and $1 \in \partial I$ to a_2 (cf. the remark in Bredon after Theorem 10.3). Moreover, we have

$$\deg(p_{b_j} f_{\partial c}) = 0$$

for $j = 1, 2$, as both maps $p_{b_j} f_{\partial c}$ are null-homotopic. The cellular chain complex is therefore

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z}^2 \xrightarrow{\partial_1} \mathbb{Z}^2 \to 0, \quad \partial_1 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} : \mathbb{Z}^2 \to \mathbb{Z}^2.$$

Both the kernel and the cokernel of ∂_1 are 1-dimensional, and therefore

$$H_k(X) \cong \begin{cases} \mathbb{Z}, & k = 0, 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

(Note that there is an even simpler CW complex structure for X with exactly one k-cell for $k = 0, 1, 2$.)

(b) $X \simeq S^2 \vee S^1$ implies $\widetilde{H}_*(X) = \widetilde{H}_*(S^2 \vee S^1) \cong \widetilde{H}_*(S^2) \oplus \widetilde{H}_*(S^1)$; hence $\widetilde{H}_2(X) = \widetilde{H}_1(X) = \mathbb{Z}$ and $\widetilde{H}_0(X) = 0$, from which the result above follows by the definition of reduced homology.

Alternatively: Excising a neighbourhood of the point joining the two spheres yields $\widetilde{H}_*(X) \cong H_*(D^2, \partial D^2) \oplus H_*(I, \partial I)$ from which the result above again follows easily.

5. We assume wlog that p and q are coprime (otherwise divide by their greatest common divisor), which implies that there exist integers a, b such that $ap - bq = 1$. Hence the matrix

$$\Psi = \begin{pmatrix} a & q \\ b & p \end{pmatrix}$$

lies in $SL(2, \mathbb{Z})$ and therefore induces a homeomorphism $\psi : T^2 \to T^2$ of $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Note that $\Psi^{-1} \in SL(2, \mathbb{Z})$ takes the line given by $px = qy$ to the line given by $x = 0$, because Ψ takes $(0, 1)$ to (q, p) (and these vectors generate the two lines). Therefore ψ^{-1} takes C to the curve C' that's the image of $x = 0$ under $\mathbb{R}^2 \to T^2$ and which is the 1-cell of the standard CW complex structure on T^2. Thus T^2/C has a CW complex structure with one cell a_k in dimensions $k = 0, 1, 2$, and the corresponding cellular differential vanishes (by the same reasons as for T^2). Therefore

$$H_k(T^2/C) \cong \begin{cases} \mathbb{Z}, & k = 0, 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

6. As discussed in class, $\mathbb{R}P^n$ has a CW complex structure with exactly one k-cell for every $k = 0, \ldots, n$. Therefore $\mathbb{R}P^n/\mathbb{R}P^m$ has a CW complex structure with one 0-cell a_0 and one k-cell a_k for every $k = m + 1, \ldots, n$. As in the case $\mathbb{R}P^n$, we have

$$\deg(p_{a_{k-1}} f_{\partial a_k}) = 1 + (-1)^k \begin{cases} 0, & k \text{ odd}, \\ 2, & k \text{ even}. \end{cases}$$

Thus the cellular chain complex $C_*(\mathbb{R}P^n/\mathbb{R}P^m)$ has one copy of $\mathbb{Z}$ in degrees $k = 0$ and $k = m+1, \ldots, n$, and the cellular differential $C_k(\mathbb{R}P^n/\mathbb{R}P^m) \to C_{k-1}(\mathbb{R}P^n/\mathbb{R}P^m)$ is $1 + (-1)^k$ for all $k = m+2, \ldots, n$ and vanishes in all other cases. The homology is therefore

$$H_k(\mathbb{R}P^n/\mathbb{R}P^m) \cong \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z}, & k = m+1 \text{ (if } m+1 \text{ is even)}, \\ \mathbb{Z}, & k = n \text{ (if } n \text{ is odd)}, \\ \mathbb{Z}_2, & m+1 \le k < n \text{ and } k \text{ odd}, \\ 0, & \text{otherwise.} \end{cases}$$