Exam Preparation

1. (a) Show that any map $f\colon \mathbb{R}^n \to X$ is homotopic to the constant map.
   (b) Are any two maps $\mathbb{R}^n \to X$ homotopic?

2. Let $f, g\colon X \to \mathbb{C} \setminus \{0\}$ be such continuous maps that $|f(x) - g(x)| < |f(x)|$ for all $x \in X$. Show that f and g are homotopic.

3. Let X be any topological space, $x_0 \in S^n$ and $f\colon S^n \to X$ a continuous map. Then the following statements are equivalent:
   (a) i. f is homotopic to some constant map c.
       ii. There exists $F\colon B^{n+1} \to X$ such that $F|_{S^n} = f$.
       iii. $f \simeq c$ (rel x_0).
   (b) Maps $f, g\colon [-1, 1] \to \mathbb{R}^2 \setminus \{(0, 0)\}$ are defined as follows:
   $$f(x) = (x, \sqrt{1 - x^2}) \quad \text{and} \quad g(x) = (x, -\sqrt{1 - x^2}).$$
   Show that $f \simeq g$, but $f \not\simeq g$ (rel $\{-1, 1\}$).

4. Which of the spaces $\mathbb{R}^2$, $\mathbb{R}^2 \setminus \{(0, 0)\}$, $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$, $\mathbb{R} \times [0, \infty)$, $S^1 \times \mathbb{R}$ and Möbius band are homotopy equivalent?

5. Let $x_0 \in X$ be a strong deformation retract of X.
   (a) Show that for any neighborhood U of x_0 there exists a neighborhood $V \subset U$ of x_0 such that the inclusion $i\colon V \hookrightarrow U$ is homotopic to a constant.
   (b) Show that X is locally path connected in x_0.

6. Let $X = (([0, 1] \times \{0\}) \cup (\{0\} \times [0, 1])) \bigcup (\cup_{n=1}^{\infty}(\{\frac{1}{n}\} \times [0, 1]))$.
   (a) Show that any point $x \in X$ is a deformation retract of X.
   (b) Find all points $x \in X$ that are strong deformation retracts of X.

7. Let
   $$T = ([0, 1] \times \{0\}) \cup (\cup_{\mathbb{Q} \cap [0,1]}(\{q\} \times [0, q])) \subset \mathbb{R} \times \mathbb{R}$$
   and let $\hat{T}$ be the set obtained by reflecting T across the line $y = x$ and translating the result by $(0, -1)$. Let $X_0 = T \cup \hat{T}$ and for $n \in \mathbb{Z}$ let X_n be the set X_0 translated by (n, n). Let $X = \cup_{n \in \mathbb{Z}} X_n$.
   (a) Show that any point $x \in X$ is a deformation retract of X.
   (b) Show that no point $x \in X$ is a strong deformation retract of X.

8. (a) Let X be contractible. Show that any $f, g\colon X \to Y$ are homotopic for any path connected space Y.
   (b) Find a non-contractible space X for which any $f, g\colon X \to Y$ are homotopic for any path connected space Y.

9. If $f_0 \sim f_1\colon X \to Y$, then $M_{f_0} \sim M_{f_1}$ rel. $X \times \{0\}$.

10. Recall that the augmentation map $\epsilon : C_0(X) \to \mathbb{Z}$ takes a 0-chain $\sum_i n_i \sigma_i$ to the integer $\sum_i n_i$. Prove that if X is non-empty and path connected then ϵ induces an isomorphism $\mathrm{H}_0(X) \to \mathbb{Z}$.

11. Find a way of identifying pairs of faces of Δ^3 (standard 3-simplex) to produce a Δ-complex structure on S^3 having a single 3-simplex, and compute the simplicial homology groups of this Δ-complex.

12. Compute the homology groups of the Δ-complex X obtained from Δ_n by identifying all faces of the same dimension. Thus X has a single k simplex for each $k \le n$.

13. Show that if A is a retract of X then the map $\mathrm{H}_n(A) \to \mathrm{H}_n(X)$ induced by the inclusion $A \subset X$ is injective.

14. Show that chain homotopy of chain maps is an equivalence relation.

15. ('3×3 lemma.') Let

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

be a commutative diagram of abelian groups. Assume that all three columns are exact. Prove that if the top two rows are exact, then so is the bottom.

16. ('Long exact homology sequence of a triple.') Let (C, B, A) be a 'triple,' so C is a space, B is a subspace of C, and A is a subspace of B. Show that there are maps $\partial\colon H_n(C, B) \to H_{n-1}(B, A)$ such that

$$\ldots \longrightarrow H_n(B, A) \xrightarrow{i_*} H_n(C, A) \xrightarrow{j_*} H_n(C, B) \to H_{n-1}(B, A) \xrightarrow{i_*} \ldots$$

is exact, where $i\colon (B, A) \to (C, A)$ and $j\colon (C, A) \to (C, B)$ are the inclusions of pairs.

17. Suppose that

$$\begin{tikzcd} \ldots \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d, "\cong"'] & A_{n-1} \arrow[r] \arrow[d] & \ldots \\ \ldots \arrow[r] & A'_n \arrow[r] & B'_n \arrow[r] & C'_n \arrow[r] & A'_{n-1} \arrow[r] & \ldots \end{tikzcd}$$

is a "ladder": a map of long exact sequences. So both rows are exact and each square commutes. Suppose also that every third vertical map is an isomorphism, as indicated. Prove that these data naturally determine a long exact sequence

$$\ldots \longrightarrow A_n \longrightarrow A'_n \oplus B_n \longrightarrow B'_n \longrightarrow A_{n-1} \longrightarrow \ldots$$

18. Let X be a quotient of a 2-sphere with the south and the north pole identified. Compute the homology groups of X.

19. Compute the homology groups of the following 2-complexes:

    (a) The space obtained from D^2 by first deleting the interiors of two disjoint subdisks in the interior of D^2 and then identifying all three resulting boundary circles together via homeomorphisms preserving clockwise orientations of these circles.

    (b) The quotient space of $S^1 \times S^1$ obtained by identifying points in the circle $S^1 \times \{x_0\}$ that differ by $2\pi/m$ rotation and identifying points in the circle $x_0 \times S^1$ that differ by $2\pi/n$ rotation.

20. Let $A \subset X$ be a retract of X. Prove that $\mathrm{H}_n(X) \cong \mathrm{H}_n(A) \oplus \mathrm{H}_n(X, A)$ for every $n \in \mathbb{N}_0$.

21. Let D be a 2-disc with k open discs removed. Compute the homology of the pair $(D, \partial D)$.

22. Compute the homology groups of the connected sum of k tori.

23. Let $f_{m,n}\colon \partial M = S^1 \to S^1 \times S^1 =: T$ be a map defined as $f_{m,n}(z) = (z^m, z^n)$, where S^1 is considered a subspace of $\mathbb{C}$, $n, m \in \mathbb{Z}$, and M denotes the Möbius band. Find the homology groups of $X_{m,n} = M \cup_{f_{m,n}} T$.

24. (a) Use the Mayer-Vietoris sequence to compute the homology groups of the space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$.

    (b) Do the same for the space obtained by attaching a Möbius band to $\mathbb{R}P^2$ via a homeomorphism of its boundary circle to the standard $\mathbb{R}P^1 \subset \mathbb{R}P^2$.

25. Show that $\tilde{H}_i(S^n \setminus X) \cong \tilde{H}_{n-i-1}(X)$ when X is homeomorphic to a finite connected graph. Hint: first do the case that the graph is a tree.

26. Consider the quotient space of a cube I^3 obtained by identifying each square face with the opposite square face via the right-handed screw motion consisting of a translation by one unit in the direction perpendicular to the face combined with a one-quarter twist of the face about its center point. Show this quotient space X is a cell complex with two 0 cells, four 1 cells, three 2 cells, and one 3 cell. Compute the homology groups of this complex.

27. Let X be the quotient space of S^2 under the identifications $x \sim -x$ for x in the equator S^1. Compute the homology groups $\tilde{H}_i(X)$. Do the same for S^3 with antipodal points of the equatorial $S^2 \subset S^3$ identified.

28. Suppose the space X is the union of open sets $A_1, \ldots, A_n$ such that each intersection $A_{i_1} \cap \ldots \cap A_{i_k}$ is either empty or has trivial reduced homology groups. Show that $\tilde{H}_i(X) = 0$ for $i \geq n - 1$, and give an example showing this inequality is best possible, for each n.

29. (a) For a finite CW complex X, the Euler characteristic $\chi(X)$ is defined to be the alternating sum $\sum_n (-1)^n c_n$, where c_n is the number is n-cells of X. Prove that $\chi(X) = \sum_n (-1)^n \mathrm{rank}\, H_n(X)$.

    (b) For finite CW complexes X and Y , show that $\chi(X \times Y) = \chi(X)\chi(Y)$.

    (c) If a finite CW complex X is the union of subcomplexes A and B, show that $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$.