We need two more definitions from homological algebra in order to define relative homology groups.

Definition

Let $\mathcal{D}=(D_\bullet, \partial_\bullet^D)$ be a chain complex.

A chain complex $\mathcal{C}=(C_\bullet, \partial_\bullet^C)$ is a **CHAIN SUBCOMPLEX** if

$C_p \subset D_p \ \forall p$ and if $\partial^C = \partial^D|_{C_\bullet}$

Definition

If $\mathcal{C} \subset \mathcal{D}$ is a subcomplex, then we can define the **QUOTIENT COMPLEX** $\mathcal{D}/\mathcal{C}$:

maps induced on quotients

$$\cdots \to D_{p+1}/C_{p+1} \xrightarrow{\partial} D_p/C_p \xrightarrow{\partial} D_{p-1}/C_{p-1} \xrightarrow{\partial} \cdots$$

$\partial \circ \partial = 0$ since the boundary maps are induced by the boundary maps of ∂.

RELATIVE HOMOLOGY

Let X be a space and $A \subset X$ a subspace. We denote by $C_n(A), C_n(X)$ the chain complexes of singular chains in A, and in X.

$C_n(A) \subset C_n(X)$ is a subcomplex.

Denote by $i : C_n(A) \to C_n(X)$ the inclusion. This is a chain map.

Let

$$C_n(X, A) := C_n(X) \big/ C_n(A),$$

where

$$S_p(X, A) = S_p(X) \big/ S_p(A)$$

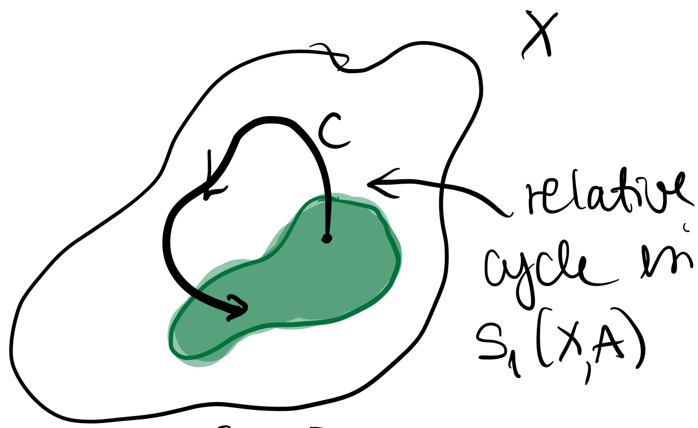
group of singular p-chains in A

So, $C_n(X, A)$ is

$$\cdots \to S_p(X)/S_p(A) \xrightarrow{\partial} S_{p-1}(X)/S_{p-1}(A) \xrightarrow{\partial} S_{p-2}(X)/S_{p-2}(A) \to \cdots$$

The homology groups of this chain complex are called **RELATIVE HOMOLOGY GROUPS**.

Intuition:



the elements of $H_p(X,A)$ are represented by **RELATIVE CYCLES** $c \in S_p(X,A)$.

A relative cycle can be represented by an n-chain $\bar{c} \in S_p(X)$ with $j(\bar{c}) = c$ such that $\partial \bar{c} \in S_{p-1}(A)$

A relative cycle c is trivial in $H_p(X,A)$ iff it is a **RELATIVE BOUNDARY**

$$c = \partial b + a \quad \text{for some } b \in S_{p+1}(X)$$

and $a \in S_p(A)$.

These properties make precise the intuitive idea that $H_p(X,A)$ is 'homology of X modulo A'.

We have the following SES of chain complexes:

$$0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X,A) \to 0.$$

This SES of chain complexes induces a LES in homology

$$\dots \xrightarrow{\partial_*} H_p(A) \xrightarrow{i_*} H_p(X) \xrightarrow{j_*} H_p(X,A) \xrightarrow{\partial_*} H_{p-1}(A) \to .$$

The connecting homomorphism has a simple description.

$$\partial_* : H_p(X,A) \to H_{p-1}(A)$$

$$\bar{c} \overset{j}{\longmapsto} c$$

$$\begin{tikzcd}
0 \arrow[r] & S_p(A) \arrow[r, "i"] \arrow[d, "\partial"] & S_p(X) \arrow[r, "j"] \arrow[d, "\partial"] & S_p(X,A) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & S_{p-1}(A) \arrow[r, "\bar{i}"] & S_{p-1}(X) \arrow[r, "j"] & S_{p-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

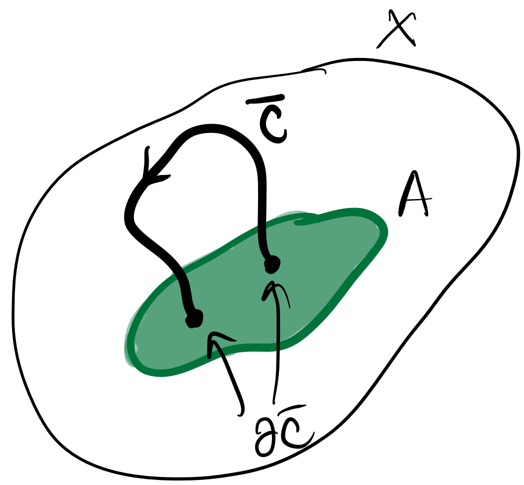
$\bar{c} \in S_p(X)$, $c \in S_p(X,A)$, $e \in S_{p-1}(A)$, $\partial\bar{c} \in S_{p-1}(X)$

$$e \longmapsto \partial \bar{c}$$

$\partial_* [c] = [e] =$

$= [\partial \bar{c}]$

the class of $\partial \bar{c}$ in A



Exactness implies that If $H_p(X,A)=0$ for all p, then the inclusion $A \hookrightarrow X$ induces isomorphisms $H_p(X) \approx H_p(A)$ $\forall p$.

So we can think of $H_p(X,A)$ as measuring the difference between the groups $H_p(X)$ and $H_p(A)$.

There is an analogous LES of reduced homology groups for a pair (X,A) with $A \neq \phi$. This comes from applying the LES theorem to $0 \to C_n(A) \to C_n(X) \to C_n(X,A) \to 0$

in non-negative dimensions,

augmented by the SES $0 \to \mathbb{Z} \xrightarrow{id} \mathbb{Z} \to 0 \to 0$.

in dimension -1.

Example

LES for (X, x_0), where $x_0 \in X$ yields

$$\cdots \cancelto{0}{H_p(x_0)} \to H_p(X) \to H_p(X, x_0) \to$$

$$\cancelto{0}{H_{p-1}(x_0)} \to H_{p-1}(X) \to \cdots$$

$\Rightarrow H_p(X, x_0) \cong H_p(X)$ for all p.

in a few weeks

Soon, we will prove the following theorem:

THEOREM

If X is a space and A is a non-empty closed subspace that is a deformation retract of some neighborhood in X, there is an exact sequence

$$\cdots \to \tilde{H}_p(A) \xrightarrow{i_*} \tilde{H}_p(X) \xrightarrow{j_*} H_p(X/A) \xrightarrow{\partial_*}$$

$$\to \tilde{H}_{p-1}(A) \xrightarrow{i_*} \tilde{H}_{p-1}(X) \xrightarrow{j_*} H_{p-1}(X/A) \to \cdots$$

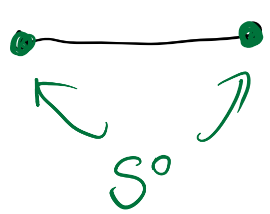
where i is the inclusion $A \hookrightarrow X$ and j is the quotient map $X \to X/A$.

EXAMPLE

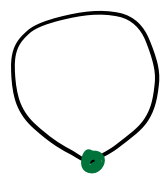
$\tilde{H}_n(S^n) \cong \mathbb{Z}$ and $\tilde{H}_p(S^n) = 0$ for $i \neq n$,

For $n > 0$ let $(X, A) = (D^n, S^{n-1})$.

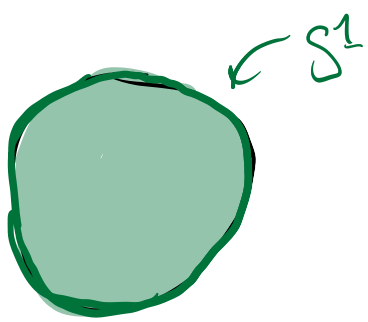
$n = 1$



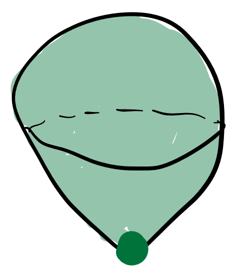
D^1/S^0



$n = 2$



D^2/S^1



For a general n, $D^n/S^{n-1} \approx S^n$.

The LES for homology for (D^n, S^{n-1}) is:

$$\cdots \tilde{H}_p(S^{n-1}) \to \cancelto{0}{\tilde{H}_p(D^n)} \to \tilde{H}_p(S^n) \xrightarrow{\cong} \tilde{H}_{p-1}(S^{n-1})$$

$$\to \cancelto{0}{\tilde{H}_{p-1}(D^n)} \to \tilde{H}_{p-1}(S^n) \to \cdots \to \tilde{H}_0(D^n) \to \tilde{H}_0(S^n) \to 0$$

D^n is contractible and therefore

$\tilde{H}_p(D^n) \cong \tilde{H}_p(\bullet) = 0 \quad \forall p.$

It follows that $\circledast\ \tilde{H}_p(S^n) \approx \tilde{H}_{p-1}(S^{n-1})$. for all p and $\tilde{H}_0(S^n) = 0$.

Let $n=0$. $S^0 \quad \bullet \quad \bullet$

We know from theorems in class

that $H_p(X) \cong \bigoplus_{\alpha \in A} H_p(X_\alpha)$, where

X_α for $\alpha \in A$ are the path-connected

components of X. So

$$H_p(S^0) = H_p(\bullet \ \bullet) \cong H_p(\bullet) \oplus H_p(\bullet)$$

$$\Rightarrow H_p(S^0) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & p=0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \tilde{H}_p(S^0) = \begin{cases} \mathbb{Z} & p=0 \\ 0 & \text{otherwise} \end{cases}$$

Now we use $(*)$ to get $\tilde{H}_p(S^1) \cong \tilde{H}_{p-1}(S^0)$ for $p > 1$ (from before we know that $\tilde{H}_0(S^1) = 0$).

$$\tilde{H}_p(S^1) = \begin{cases} \mathbb{Z} & p=1 \\ 0 & \text{otherwise} \end{cases}$$

By induction it follows that $\tilde{H}_p(S^n) = \begin{cases} \mathbb{Z} & p=n \\ 0 & \text{otherwise} \end{cases}$

If we have a map

$$f : (X, A) \to (Y, B), \text{ then}$$

we also have an induced map

$$f_c : C_n(X, A) \to C_n(Y, B)$$

& $$f_* : H_p(X, A) \to H_p(Y, B) \quad \forall p.$$

(Since f_c takes $C_n(A)$ to $C_n(B)$ the map on quotients is well defined. Also, $f_c \partial = \partial f_c$ holds for relative chains since it holds for absolute chains). We have the following statement about homotopy invariance

PROPOSITION

If two maps $f, g : (X, A) \to (Y, B)$ are homotopic through maps of pairs $(X, A) \to (Y, B)$, then

$$f_* = g_* : H_n(X, A) \to H_n(Y, B).$$

Proof

Exercise (proof in Hatcher on page 118).

Finally, consider $B \subset A \subset X$.

We have a SES of chain complexes

$$0 \to C_n(A, B) \to C_n(X, B) \to C_n(X, A) \to 0.$$

This sequence induces a LES

$$\cdots H_n(A,B) \to H_n(X,B) \to H_n(X,A) \to$$
$$\to H_{n-1}(A,B) \to \cdots$$

SPLIT EXACT SEQUENCES

Let $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ be a SES of abelian groups.

Definition

The sequence is called **SPLIT** if $\exists$ an isomorphism $\tau : B \xrightarrow{\cong} A \oplus C$ s.t. the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{i} & B & \xrightarrow{j} & C & \to & 0 \\ & & \downarrow \mathrm{id} & & \tau \downarrow \cong & & \downarrow \mathrm{id} & & \\ 0 & \to & A & \xrightarrow[i_A]{} & A \oplus C & \xrightarrow[\pi_C]{} & C & \to & 0 \end{array}$$

where $i_A(a) := (a, 0)$ and $\pi_C(a,c) := c$.

Proposition

To say that the SES $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is split is equivalent to any of the following three statements

① $\exists$ a homomorphism $e : B \to B$ with $e \circ e = e$, s.t $\ker e = \operatorname{im} i$.

② $\exists$ a homomorphism $C \xrightarrow{s} B$ s.t. $j \circ s = id_C$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & B \arrow[r, "j"] & C \arrow[r] \arrow[l, bend left, "s"] & 0 \end{tikzcd}$$

(s is a right inverse to j)

③ $\exists$ a left inverse to i, i.e. a homomorphism $u : B \to A$ with $u \circ i = id_A$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r] \arrow[l, bend left, "u"] & C \arrow[r] & 0 \end{tikzcd}$$

Proof

split $\Rightarrow$ (1):

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "id"] & B \arrow[r, "j"] \arrow[d, "\tau"] & C \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & A \arrow[r, "i_A"'] & A \oplus C \arrow[r] & C \arrow[r] \arrow[l, bend left, "i_C"] & 0
\end{tikzcd}$$

Put $e(b) := \tau^{-1} \circ i_C \circ j(b)$

Exercise: check that $e \circ e = e$ and $\ker e = \operatorname{im} i$.

(1) $\Rightarrow$ split

Note that $b - e(b) \in \operatorname{Im} i$:

$$e(b - e(b)) = e(b) - e \circ e(b) =$$
$$= e(b) - e(b) = 0$$

$\Rightarrow b - e(b) \in \ker e = \operatorname{Im}(i)$.

Define $t(b) := (a, j(b))$, where $a \in A$ is the unique element with

$i(a) = b - e(b)$.

Exercise: Check that τ is a homomorphism and that it makes the diagram in the definition commutative

$2 \Rightarrow 1$

Put $e(b) := s \circ j(b)$.

split $\Rightarrow 2$

Put $s(c) := \tau^{-1} \circ i_C(c)$.

$3 \Leftrightarrow 1/\text{split}$

Exercise

$\boxtimes$

Example

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{j} \mathbb{Z}/2\mathbb{Z} \to 0$$

is a non-split SES, because $\mathbb{Z}$ is not isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

(Alternatively, $\nexists\ \mathbb{Z} \xleftarrow{s} \mathbb{Z}/2\mathbb{Z}$ right inverse to j because s must be 0.)

PROPOSITION

Let $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ be a SES of abelian groups. If C is a free abelian, then the sequence splits.

Proof

Let $\{c_\alpha\}_{\alpha \in I}$ be a basis for C.

Define $s: C \to B$ as follows:

$\forall \alpha \in I$ pick $b_\alpha \in j^{-1}(c_\alpha) \subset B$.

not empty since j is surjective

Define $s(c_\alpha) := b_\alpha$.

Now extend linearly to $s: C \to B$. Clearly, $j \circ s = id_C$ $\blacksquare$

Let $A_\bullet, B_\bullet, \mathcal{C}_\bullet$ be chain complexes, and $0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} \mathcal{C}_\bullet \to 0$ be a SES of chain complexes. The sequence is called **SPLIT** (in the sense of chain complexes) if $\exists$ a **CHAIN MAP** $s: \mathcal{C}_\bullet \to B_\bullet$ with $j \circ s = id_{\mathcal{C}_\bullet}$.

$\iff$ $\exists$ a splitting τ with τ being a chain map

$\iff$ $\exists$ u left-inverse of i with u = chain map.

If $\mathcal{A}_\bullet, \mathcal{C}_\bullet$ are chain complexes we can define $\mathcal{A}_\bullet \oplus \mathcal{C}_\bullet$, where

$$(\mathcal{A}_\bullet \oplus \mathcal{C}_\bullet)_p = A_p \oplus C_p$$

and the boundary operator is

$$\partial := \partial^A \oplus \partial^B$$

Note that $H_p(\mathcal{A}_\bullet \oplus \mathcal{C}_\bullet) \cong H_p(\mathcal{A}_\bullet) \oplus H_p(\mathcal{C}_\bullet)$

because $Z_p(\mathcal{A}_\bullet \oplus \mathcal{C}_\bullet) = Z_p(\mathcal{A}_\bullet) \oplus Z_p(\mathcal{C}_\bullet)$
$B_p(\mathcal{A}_\bullet \oplus \mathcal{C}_\bullet) = B_p(\mathcal{A}_\bullet) \oplus B_p(\mathcal{C}_\bullet)$.

If $0 \to \mathcal{A}_\bullet \to \mathcal{B}_\bullet \to \mathcal{C}_\bullet \to 0$ is a split SES of complexes, then

$$H_p(B_\bullet) \cong H_p(A_\bullet) \oplus H_p(C_\bullet) \ \forall p.$$

Remark

Sometimes $0 \to A_p \to B_p \to C_p \to 0$ splits for all p as a sequence of abelian groups, but NOT as a sequence of chain complexes.

Example

X = space, $A \subseteq$ subspace

$$0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X, A) \to 0$$

Claim: $\forall p$, $S_p(X, A)$ is free abelian.

A basis for this group: consider

$$\{\sigma : \Delta^p \to X : \sigma(\Delta^p) \not\subseteq A\} =: \xi$$

and $\{j(\sigma)\}_{\sigma \in \xi}$. This family freely generates $S_p(X, A)$.

So, $\forall p$, the sequence

$$0 \to S_p(A) \to S_p(X) \to S_p(X,A) \to 0$$

splits as a sequence of abelian groups, but usually NOT as chain complexes since usually this splitting is not a chain map.

Usually $H_p(X) \not\cong H_p(A) \oplus H_p(X,A)$.

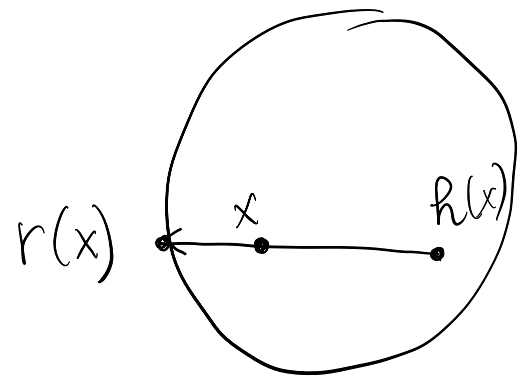
EXAMPLE

Brouwer fixed point theorem

Every continuous map $h: D^n \to D^n$ has a fixed point, that is, a point $x \in D^n$ with $h(x) = x$.

Suppose that $h(x) \neq x \quad \forall x \in D^n$ (proof by contradiction).

then we can define $r: D^n \to S^{n-1}$ by letting $r(x)$ be the point of S^{n-1} where the ray in $\mathbb{R}^n$ starting at $h(x)$ and passing through x leaves D^n.



This map is continuous & $r(x) = x \in S^{n-1}$, or with other words, a retraction.

For $A = S^{n-1}$, $X = D^n$ such an $r: X \to A$ gives a splitting

$$H_p(D^n) \cong H_p(S^{n-1}) \oplus H_p(D^n, S^{n-1}).$$

However, for $p = n-1$

$$H_{n-1}(D^n) = 0,$$

whereas $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$, which is not possible.