EXCISION

A fundamental property of relative homology groups is given by the following **EXCISION THEOREM**, describing when the relative groups $H_n(X,A)$ are unaffected by excising/deleting a subset $Z \subset A$.

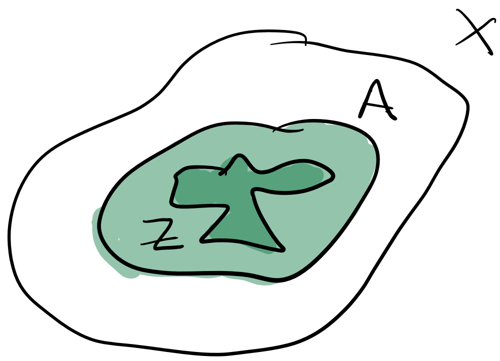
THEOREM (EXCISION)

Given subspaces $Z \subset A \subset X$ such that the closure of Z is contained in the interior of A, then the inclusion $(X-Z, A-Z) \hookrightarrow (X,A)$ induces isomorphisms $H_p(X-Z, A-Z) \to H_p(X,A)$ for all p.

Equivalently, for subspaces $A, B \subset X$ whose interior covers X, the inclusion $(B, A \cap B) \hookrightarrow (X,A)$ induces isomorphisms

$H_p(B, A\cap B) \to H_p(X, A)$ for all p.

The translation between the two versions is obtained by setting



$$B = X - Z \quad \& \quad Z = X - B.$$

Then $A\cap B = A - Z$ and the condition $\overline{Z} \subset \mathring{A}$ is equivalent to

$$X = \mathring{A} \cup \mathring{B} \text{ since } X - \mathring{B} = \overline{Z}.$$

The proof is quite technical and will be done in several steps.

RELATING HOMOLOGY GROUPS OF A COVERING TO HOMOLOGY GROUPS OF A SPACE

Let X be a space and $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be a collection of subsets of X s.t.

the interiors of the $\mathcal{U}_\alpha$'s cover X,

$$X = \bigcup_{\alpha \in \mathcal{A}} \mathring{\mathcal{U}}_\alpha .$$

We say that a subset $Q \subset X$ is **$\mathcal{U}$-small** if $\exists \alpha \in \mathcal{A}$ s.t. $Q \subset \mathcal{U}_\alpha$.

Consider the subgroup of $S_p(X)$ generated by $S_p(\mathcal{U}_\alpha)$ $\forall \alpha$. Denote it by $S_p^{\mathcal{U}}(X)$. The elements are chains $\sum_i n_i \sigma_i$ such that each σ_i has image contained in some set in the cover $\mathcal{U}$.

The boundary map $\partial : S_p(X) \to S_{p-1}(X)$ takes $S_p^{\mathcal{U}}(X)$ to $S_{p-1}^{\mathcal{U}}(X)$, so the groups $S_p^{\mathcal{U}}(X)$ form a chain complex.

We denote this chain complex $C_n^{\mathcal{U}}(X)$ and it is a subcomplex of $C_n(X)$.

We denote the homology groups of $C_n^{\mathcal{U}}(X)$ by $H_p^{\mathcal{U}}(X)$.

THEOREM 1

The inclusion chain map $i^{\mathcal{U}}: C_n^{\mathcal{U}}(X) \to C_n(X)$ induces an isomorphism in homology

$$i_*^{\mathcal{U}} : H_p^{\mathcal{U}}(X) \xrightarrow{\cong} H_p(X) \quad \forall p.$$

To prove theorem 1, we will apply the so-called barycentric subdivision process.

BARYCENTRIC SUBDIVISION

1 BARYCENTRIC SUBDIVISION OF SIMPLICES

Let $\sigma = [v_0, v_1, \dots, v_n]$ be an n-simplex in $\mathbb{R}^d$. Then

$$\sigma = \left\{ \sum_{i=0}^{n} t_i v_i \;\middle|\; 0 \leq t_i \leq 1, \sum t_i = 1 \right\}$$

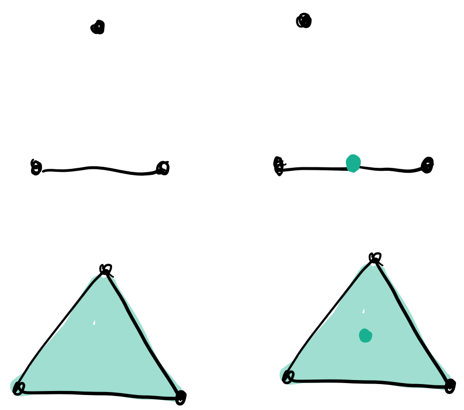
The **BARYCENTER** or 'center of gravity' of the simplex σ is the point

$$b = b_\sigma = \frac{1}{n+1} \sum_{i=0}^{n} v_i$$

$n=0$

$n=1$

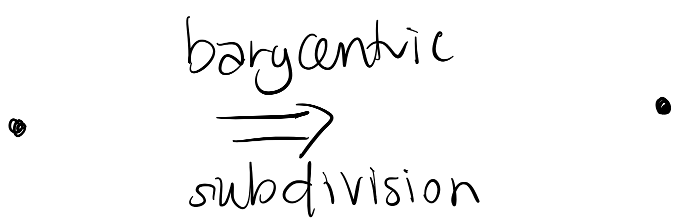
$n=2$



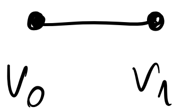
The **BARYCENTRIC SUBDIVISION** of $[v_0, v_1, \dots, v_n]$ is the decomposition of $[v_0, v_1, \dots, v_n]$ into n-simplices $[b, w_0, \dots, w_{n-1}]$ where, inductively, $[w_0, \dots, w_{n-1}]$ is an $(n-1)$-simplex in the barycentric subdivision of a face $[v_0, \dots, \hat{v}_i, \dots, v_n]$.

The induction starts with $n=0$.

$n=0$



$n=1$

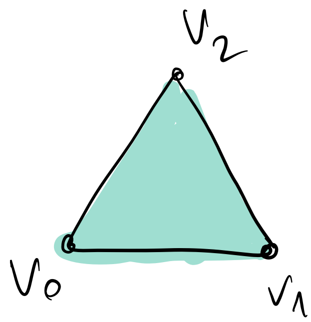


$\xrightarrow[\text{subdivision}]{\text{barycentric}}$

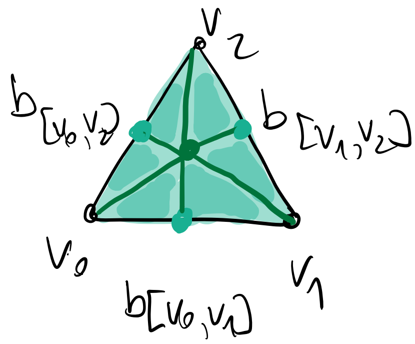
$[b, v_0]$

$[b, v_1]$

$n=2$



$\xrightarrow[\text{subdivision}]{\text{barycentric}}$



$n \geq 2$

$\sigma = [v_0, \dots, v_n] \xrightarrow[\text{subdivision}]{\text{barycentric}}$ n-simplices

$\tau = [b, w_0, \dots, w_{n-1}]$

where $[w_0, \dots, w_{n-1}]$ is a $(n-1)$-sx from the barycentric subdivision of a face $[v_0, \dots, \hat{v}_i, \dots, v_n]$ of σ

CLAIM

$$\operatorname{diam}[b, w_0, \dots, w_{n-1}] \leq \frac{n}{n+1} \operatorname{diam}[v_0, \dots, v_n]$$

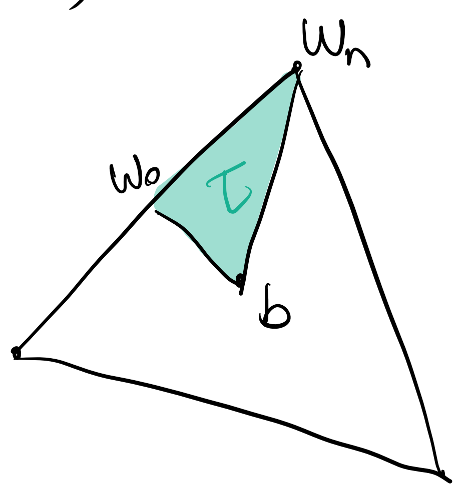
max distance between any two of its vertices since

$$\left|v - \sum_i t_i v_i\right| = \left|\sum_i t_i (v - v_i)\right| \leq \sum_i t_i |v - v_i|$$

$$\leq \sum_i t_i \max_j |v-v_j| = \max_j |v-v_j|$$

To obtain the bound, we therefore need to verify that the distance between any two vertices w_j and w_k of a simplex τ of the barycentric subdivision of $[v_0,\dots,v_n]$ is at most

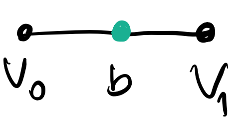
$$\frac{n}{(n+1)} \operatorname{diam}[v_0,\dots,v_n].$$



① w_j & $w_k \neq b$, the barycenter of $[v_0,\dots,v_n]$. In this case the statement follows by induction on n as these two points lie in a proper face of $[v_0,\dots,v_n]$:

$$\operatorname{diam}[v_0,b] = \operatorname{diam}[v_1,b] \leq \frac{1}{2}\operatorname{diam}[v_0,v_1]$$

$n=1$



$$|w_i - w_j| \overset{IH}{\leq} \frac{n-1}{n} \operatorname{diam}[v_0, \dots, \hat{v}_i, \dots, v_n]$$

$$\leq \frac{n}{n+1} \operatorname{diam}[v_0, \dots, v_n]$$

$$\frac{n-1}{n} \leq \frac{n}{n+1}$$

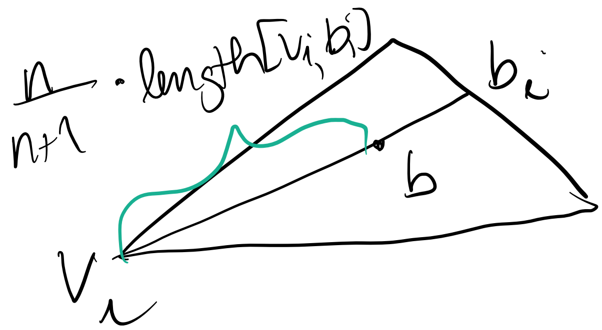
(2) Suppose WLOG that $w_j = b$.

then

$|b - w_k| \leq |b - v_i|$ for some i.

Let b_i be the barycenter of $[v_0, \dots, \hat{v}_i, \dots, v_n]$

$$b_i := \frac{1}{n} \sum_{j \neq i} v_j$$



$$b = \frac{1}{n+1} v_i + \frac{n}{n+1} b_i$$

$$|b - v_i| = \frac{n}{n+1} |b_i - v_i|$$

$$\leq \frac{n}{n+1} \operatorname{diam}[v_0, \dots, v_n]$$

IMPORTANT: $\left(\frac{n}{n+1}\right)^r \xrightarrow[r\to\infty]{} 0$

② BARYCENTRIC SUBDIVISION OF LINEAR CHAINS

Let $Y \subset \mathbb{R}^d$ be a convex set.

We define

$$LS_p(Y) = \langle \sigma : \Delta^p \to Y \mid \sigma \text{ is a linear map} \rangle$$

$\sigma\left(\sum_{i=0}^{p} t_i e_i\right) = \sum_{i=0}^{p} t_i \sigma(e_i)$

linear simplices in Y

standard basis

$LS_p(Y) \subset S_p(Y)$ & the boundary map maps $LS_p(Y)$ to $LS_{p-1}(Y)$.

Let $LS_{-1}(Y) = \mathbb{Z}\langle[\phi]\rangle$ ← empty simplex

and $\partial[w_0] = [\phi]$ $\forall$ 0-sx w_0.

We have the following chain complex,

$$\cdots \to LS_p(Y) \to LS_{p-1}(Y) \to \cdots \to LS_1(Y) \to LS_0(Y) \to \mathbb{Z} \to \cdots$$

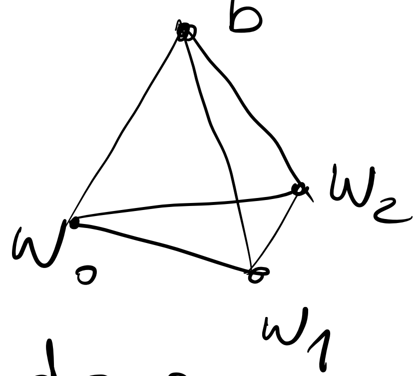
a subcomplex of $C_n(Y)$ that we denote by $LC_n(Y)$.

Each $b \in Y$ determines a homomorphism $b : LS_p(Y) \to LS_{p+1}(Y)$ defined by:

$$b([w_0, \dots, w_p]) = [b, w_0, \dots, w_p]$$

& extended to all of $LS_p(Y)$ linearly.

CONE OPERATOR



b sends a linear chain to the cone that has this chain as a base & whose tip is b

Let's compute

$$\partial\left(b[w_0,\dots,w_p]\right) = \partial\left([b,w_0,\dots,w_p]\right)$$

$$= (-1)^0[w_0,\dots,w_p] + (-1)^1[b,w_1,\dots,w_p]$$

$$+ (-1)^2[b,w_0,w_2,\dots,w_p] + \dots + (-1)^p[b,w_0,\dots,w_{p-1}]$$

$$= [w_0,\dots,w_p] - b\big([w_1,\dots,w_p] + (-1)^1[w_0,w_2,\dots,w_p]$$

$$+ \dots + (-1)^p[w_0,\dots,w_{p-1}]\big) =$$

$$= [w_0,\dots,w_p] - b\,\partial[w_0,\dots,w_p] =$$

$$= (id - b\circ\partial)[w_0,\dots,w_p]$$

$\Rightarrow$ $\partial b = id - b\circ\partial$

b is a **CHAIN HOMOTOPY** between 0 and the identity on the augmented chain complex $LC_n(I)$.

Now we define a **SUBDIVISION HOMOMORPHISM** $sd_p : LS_p(Y) \to LS_p(Y)$ by induction on p.

$p=-1$

$$sd_{-1}([\phi]) = [\phi]$$

$$sd_{-1} = id$$

$p \geq 0$ for generators $\sigma \in LS_p(Y)$

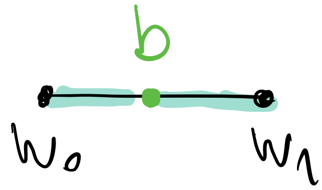
$$sd_p(\sigma) = b_\sigma(sd_{p-1}(\partial\sigma)),$$

where b_σ is the barycenter of σ

$p=0$ $\quad sd_0([w_0]) = w_0([\phi]) = [w_0]$

$$\Rightarrow sd_0 = id$$

$p=1$



$$sd_1([w_0,w_1]) =$$

$$= b\left((-1)^0[w_1] + (-1)^1[w_0]\right)$$

$$= b([w_1]-[w_0]) =$$

$= [b, w_1] - [b, w_0]$

sum of the 1-simplices in the barycentric subdivision with certain signs

(compare the def. of sd with that of the subdivision of a simplex)

sd is a chain map.

We prove this by induction:

$$\underset{\text{id}}{\underset{\parallel}{sd_{-1}}} \circ \partial = \partial \circ \underset{\text{id}}{\underset{\parallel}{sd_0}} \qquad \checkmark$$

$$\begin{tikzcd}
\cdots \arrow[r] & LS_{p+1}(Y) \arrow[r] \arrow[d, "sd_{p+1}"] & LS_p(Y) \arrow[r] \arrow[d, "sd_p"] & \cdots \arrow[r] & LS_0(Y) \arrow[r] \arrow[d, "id"] & LS_{-1}(Y) \arrow[r] \arrow[d, "id"] & 0 \\
\cdots \arrow[r] & LS_{p+1}(Y) \arrow[r] & LS_p(Y) \arrow[r] & \cdots \arrow[r] & LS_0(Y) \arrow[r] & LS_{-1}(Y) \arrow[r] & 0
\end{tikzcd}$$

Squares up to p commute.

$\partial b_\sigma = id - b_\sigma \partial$

induction step:

$\partial(sd_{p+1}(\sigma)) = \partial(b_\sigma(sd_p(\partial\sigma))) =$

$\partial sd_p = sd_{p-1}\partial$

$= sd_p(\partial\sigma) - b_\sigma(\partial sd_p(\partial\sigma)) =$

$\partial\partial = 0$

$= sd_p(\partial\sigma) - b_\sigma(\cancel{sd_{p-1}\partial(\partial\sigma)}) \xrightarrow{} 0 =$

$= sd_p(\partial\sigma)$

Next we build a chain homotopy $D: LS_p(Y) \to LS_{p+1}(Y)$ between sd and id.

chain map

$D_p : LS_p(Y) \to LS_{p+1}(Y)$

s.t. $\partial D + D\partial = id - sd$

We define D inductively.

$$\begin{tikzcd}
\cdots \arrow[r] & LS_{p+1}(Y) \arrow[r] & LS_p(Y) \arrow[r] & \cdots \arrow[r] & LS_0(Y) \arrow[r] \arrow[d, "sd"', "id"] & LS_{-1}(Y) \arrow[r] \arrow[d, "sd"', "id"] \arrow[dl, "D_{-1}"'] & 0 \arrow[dl, "D_{-2}"] \\
\cdots \arrow[r] & LS_{p+1}(Y) \arrow[r] & LS_p(Y) \arrow[r] & \cdots \arrow[r] & LS_0(Y) \arrow[r] & LS_{-1}(Y) \arrow[r] & 0
\end{tikzcd}$$

$D_{-2} = 0$

$D_{-1} = 0$

$$\partial D_{-1} + D_2 \partial = 0 + 0 = 0$$

$$id - sd_{-1} = id - id = 0$$

$$\Rightarrow \partial D_{-1} + D_{-2}\partial = id - sd_{-1}$$

$p \geq 0$ We define D_p inductively.

$\sigma \in LS_p(Y)$ a simplex

$$D_p(\sigma) = b_\sigma\left(\sigma - D_{p-1}(\partial\sigma)\right)$$

barycenter of σ

We check using induction that $\partial D + D\partial = id - sd$. Assume that all maps up to D_p satisfy this.

$$\partial D_{p+1}\sigma = \partial\big(b_\sigma(\sigma - D_p(\partial\sigma))\big) \;\overset{\partial b_\sigma = id - b_\sigma \partial}{=}$$

$$= \sigma - D_p(\partial\sigma) - b_\sigma\big(\partial(\sigma - D_p(\partial\sigma))\big)$$

$$= \sigma - D_p(\partial\sigma) - b_\sigma(\partial\sigma - \partial D_p(\partial\sigma))$$

$$\begin{tikzcd}
\cdots \arrow[r] & LS_{p+2}(Y) \arrow[r] \arrow[d] & LS_{p+1}(Y) \arrow[r, "\partial"] \arrow[d, "\substack{sd\\id}"] \arrow[dl, "D_{p+1}"'] & LS_p(Y) \arrow[r] \arrow[d, "\substack{sd\\id}"] \arrow[dl, "D_p"'] & LS_{p-1}(Y) \arrow[r] \arrow[d] \arrow[dl, "D_{p-1}"'] & \cdots \\
\cdots \arrow[r] & LS_{p+2}(Y) \arrow[r, "\partial"'] & LS_{p+1}(Y) \arrow[r] & LS_p(Y) \arrow[r] & LS_{p-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

IH: $\partial D_p + D_{p-1}\partial = id - sd$

$\Rightarrow id - \partial D_p = sd_p + D_{p-1}\partial$

$$\overset{I.H}{=} \sigma - D_p(\partial\sigma) - b_\sigma\big(sd_p(\partial\sigma) + D_{p-1}\partial(\partial\sigma)\big)$$

$$= \sigma - D_p\partial(\sigma) - b_\sigma\, sd_p(\partial\sigma)$$

$$= \sigma - D_p\partial(\sigma) - sd_{p+1}(\sigma)$$

$$\Rightarrow \partial D_{p+1}(\sigma) + D_p\partial(\sigma) = \sigma - sd_{p+1}(\sigma)$$

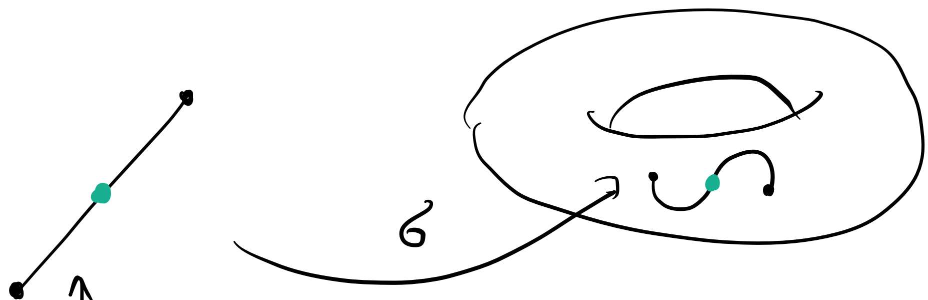
③ BARYCENTRIC SUBDIVISION OF GENERAL CHAINS

Let X be a topological space.

Define homomorphisms

$$sd = sd_p : S_p(X) \to S_p(X)$$

on generators $\sigma(\sigma : \Delta^p \to X)$



We subdivide this space, which is a convex subset in $\mathbb{R}^{p+1}$

$$sd(\sigma) = \sigma_c\big(sd(\underbrace{id : \Delta^p \to \Delta^p})\big)$$

$$(id : \Delta^p \to \Delta^p) \in LS_p(\Delta^p) \subset S_p(\Delta^n)$$

$\sigma : \Delta^p \to X$ induces $\sigma_c : S_p(\Delta^p) \to S_p(X)$.

$$\begin{tikzcd} \underset{\mathrm{id}\,\in}{S_p(\Delta^p)} \arrow[r, "\sigma_c"] \arrow[d, "sd"] & \underset{\ni\,\sigma}{S_p(X)} \arrow[d, "sd"] \\ S_p(\Delta^p) \arrow[r, "\sigma_c"] & S_p(X) \end{tikzcd}$$

$S_p(\Delta^p) \ni sd(id)$

sd is a chain map

$$\partial(sd(\sigma)) = \partial\, \sigma_c(sd(id : \Delta^p \to \Delta^p)) =$$

$$= \sigma_c\, \partial(sd(id : \Delta^p \to \Delta^p)) =$$

σ_c is a chain map

$$= \sigma_c\, sd(\partial\, id_{\Delta^p}) =$$

$$= \sigma_c\, sd\left(\sum_{i=0}^{p} (-1)^i id_{\Delta_i^p}\right)$$

restriction of id to the ith face of Δ^p

$$= \sum_{i=0}^{p} (-1)^i \sigma_c \, sd(id_{\Delta_i^p})$$

sum of signed simplices in the barycentric subdivision of Δ_i^p

$$= \sum_{i=0}^{p} (-1)^i \, sd\left(\sigma_c\big|_{\Delta_i^p}\right)$$

$$= sd\left(\sum_{i=0}^{p} (-1)^i \sigma_c\big|_{\Delta_i^p}\right) =$$

$$= sd(\partial\sigma)$$

In a similar fashion we define

$D : S_p(X) \to S_{p+1}(X)$

$D(\sigma) = \sigma_c(D(id_{\Delta^p}))$

here we take the D defined for linear chains

D is a chain homotopy between sd & id.

$$\partial D(\sigma) = \partial(\sigma_c(D(id_{\Delta^p}))) =$$

σ_c is a chain map

$$= \sigma_c(\partial D(id_{\Delta^p})) =$$

D is a chain homotopy for linear chains $\rightarrow = \sigma_c \left(id_{\Delta^p} - sd(id_{\Delta^p}) - D\partial(id_{\Delta^p}) \right)$

$= \sigma - sd(\sigma) - D\partial(\sigma) =$

$= (id - sd - D\partial)(\sigma)$

PROOF OF THEOREM 1

Let $\mathcal{U}$ be a covering as in the statement of Theorem 1. Let $\sigma \in S_p(X)$ be a singular simplex.

Then $\{ \sigma^{-1}(\mathring{U}) \mid U \in \mathcal{U} \}$ is an open covering of Δ^p. Δ^p is compact, so we can select the Lebesgue number δ of this covering

[Lebesgue's number Lemma :

If the metric space (X,d) is compact & an open cover of X is given, then there exists a number $\delta > 0$ such that every subset of X having diameter less than δ is contained in some member of the cover.]

Pick $m \in \mathbb{N}$ large enough that

diameter of an n-simplex

$$\left(\frac{n}{n+1}\right)^{m_\delta} \sqrt{2} \leq \delta.$$

m will determine how much we have to subdivide simplices so that each lies in some $U \in \mathcal{U}$

If we use sd m-times on

σ we get a chain consisting of singular simplices, of which each lies in some $U \in \mathcal{U}$.

$\sigma_c(sd^m(id_{\Delta^p})) = sd^m(\sigma) \in S_p^{\mathcal{U}}(X)$.

For each p-simplex σ we select the m_σ in a way that it is the smallest non-negative integer for which $sd^{m_\sigma}(\sigma) \in S_p^{\mathcal{U}}(X)$

$(m_\sigma = 0 \Leftarrow \neg\, \sigma \in S_p^{\mathcal{U}}(X))$.

We define

$$\overline{D} : S_p(X) \to S_{p+1}(X)$$

$$\overline{D}(\sigma) = \sum_{j=0}^{m_\sigma - 1} D(sd^j(\sigma))$$

for σ a p-simplex

this is the D that we defined for linear chains

If $m_\sigma = 0$, $\overline{D}(\sigma) = 0$.

We calculate

$$(\partial \overline{D} + \overline{D} \partial)(\sigma) = \partial \sum_{j=0}^{m_\sigma - 1} D(sd^j(\sigma)) + \sum_{i=0}^{p} (-1)^i \overline{D} \sigma^i =$$

i-th face of σ

$$= \sum_{j=0}^{m_\sigma - 1} \partial D(sd^j(\sigma)) + \sum_{i=0}^{p} (-1)^i \sum_{j=0}^{m_{\sigma_i} - 1} D(sd^j(\sigma^i))$$

$$= \sum_{j=0}^{m_\sigma - 1} \left(sd^j(\sigma) - sd^{j+1}(\sigma) - D\partial(sd^j(\sigma)) \right)$$

$$+ \sum_{i=0}^{p} (-1)^i \sum_{j=0}^{m_{\sigma_i} - 1} D(sd^j(\sigma^i)) =$$

$$= \sigma - sd^{m_\sigma}(\sigma) - \sum_{j=0}^{m_\sigma - 1} D(sd^j(\partial\sigma)) +$$

$$+\sum_{i=0}^{p}(-1)^i\sum_{j=0}^{m_{\sigma_i}-1}D(sd^j(\sigma^i))=$$

$$=\sigma-sd^{m_\sigma}(\sigma)-\sum_{j=0}^{m_\sigma-1}\sum_{i=0}^{p}(-1)^i D(sd^j(\sigma_i))$$

$$+\sum_{i=0}^{p}(-1)^i\sum_{j=0}^{m_{\sigma_i}-1}D(sd^j(\sigma^i))=$$

$$=\sigma-sd^{m_\sigma}(\sigma)+\sum_{i=0}^{p}(-1)^i\sum_{j=m_{\sigma_i}}^{m_\sigma-1}D(sd^j(\sigma^i))$$

$$(m_{\sigma_i}\le m_\sigma)$$

We set

$$\rho(\sigma):=\sigma-\partial\overline{D}(\sigma)-\overline{D}\partial(\sigma)$$

Note that $\rho(\sigma)\in S_p^{\mathcal{U}}(X)$.

This ρ is a map: $S_p(X)\to S_p^{\mathcal{U}}(X)$.

ρ is a chain map:

$$\partial\rho(\sigma) = \partial\sigma - \partial\partial\bar{D}(\sigma) - \partial\bar{D}\partial(\sigma)$$

$$= \partial\sigma - \partial\bar{D}\partial(\sigma)$$

$$= \partial\sigma - \partial\bar{D}\partial(\sigma) - \bar{D}\partial\partial(\sigma)$$

$$= \rho(\partial\sigma)$$

$\Rightarrow \partial\bar{D} - \bar{D}\partial = id - i^u_c \circ \rho$,

where $i^u_c : C_n^u(X) \to C_n(X)$ is the inclusion.

$\bar{D}$ is a chain homotopy from $i^u_c \circ \rho$ to id.

Also, $\rho \circ i^u_c = \rho(i^u_c(\sigma)) =$

$$= \sigma - \overbrace{\partial\bar{D}(i^u_c(\sigma))}^{=0} - \overbrace{\bar{D}\partial(i^u_c(\sigma))}^{=0}$$

$= id$

so ρ is the chain homotopy inverse of $i_c^{\mathcal{U}}$.

It follows from homotopy invariance statements that $i_*^{\mathcal{U}}$ is an isomorphism $H_p^{\mathcal{U}}(X) \xrightarrow{i_*^{\mathcal{U}}} H_p(X)$.

PROOF OF EXCISION THEOREM

Let $\mathcal{U} = \{A, B\}$ such that $\mathring{A} \cup \mathring{B} = X$,

$$i_c^{\mathcal{U}} : C_n^{\mathcal{U}}(X) \to C_n(X)$$

is a chain equivalence. From proof of theorem 1 we get maps ρ & $\overline{D}$ that map simplices in A to simplices in A.

ρ and $\overline{D}$ induce maps on quotients

$$\rho : S_p(X)/S_p(A) \to S_p^{\mathcal{U}}(X)/S_p(A)$$

$$\overline{D} : S_p(X)/S_p(A) \to S_{p+1}(X)/S_{p+1}(A)$$

It still holds that

$$\partial\overline{D} + \overline{D}\partial = id - i_c^{\mathcal{U}} \circ \rho$$

and that

$$i_c^{\mathcal{U}} : C_n^{\mathcal{U}}(X)/C_n(A) \to C_n(X)/C_n(A)$$

is a chain equivalence and consequently it induces an isomorphism on homology.

The map

$$S_p(B) \big/ S_p(A \cap B) \longrightarrow S_p^{\mathcal{U}}(X) \big/ S_p(A)$$

induced by inclusion is an isomorphism since both quotient groups are free with the basis singular p-simplices in B that do not lie in A. $\Rightarrow$

$$H_p(X, A) \cong H_p\left(C_n^{\mathcal{U}}(X) \big/ C_n(A) \right)$$

$$\cong H_p(B, A \cap B).$$

$\blacksquare$

Here is an example of the machinery we developed, a classical result from 1910 due to Brouwer, known as

INVARIANCE OF DIMENSION

If non-empty open sets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$.

Let $x \in U$. By excision

$$H_p(U, U - \{x\}) \cong H_p(\mathbb{R}^m, \mathbb{R}^m - \{x\}).$$

From LES of $(\mathbb{R}^m, \mathbb{R}^m - \{x\})$

$$\cdots \tilde{H}_p(\mathbb{R}^m - \{x\}) \to \tilde{H}_p(\mathbb{R}^m) \to H_p(\mathbb{R}^m, \mathbb{R}^m - \{x\}) \to$$
$$\to \tilde{H}_{p-1}(\mathbb{R}^m - \{x\}) \to \tilde{H}_{p-1}(\mathbb{R}^m) -$$

we get $\quad H_p(\mathbb{R}^m, \mathbb{R}^m - \{x\}) \cong \tilde{H}_{p-1}(\mathbb{R}^m - \{x\})$

Since $\mathbb{R}^m - \{x\}$ strongly deformation retracts to S^{m-1},

$$H_p(U, U - \{x\}) \cong H_{p-1}(S^{m-1}) = \begin{cases} \mathbb{Z} & p = m \\ 0 & \text{otherwise} \end{cases}$$

Homeomorphism $h: U \to V$ yields a homeomorphism of pairs

$$(U, U - \{x\}) \text{ and } (V, V - \{h(x)\})$$

and so

$$H_p(U, U-\{x\}) \cong H_p(V, V-\{h(x)\}).$$

Since also

$$H_p(V, V-\{h(x)\}) \cong H_{p-1}(S^{n-1}) = \begin{cases} \mathbb{Z} & p=n \\ 0 & \text{otherwise} \end{cases}$$

it follows that $m=n$.