We have the following chain complex,

$$\cdots \to LS_p(Y) \to LS_{p-1}(Y) \to \cdots \to LS_1(Y) \to LS_0(Y) \to \mathbb{Z} \to \cdots$$

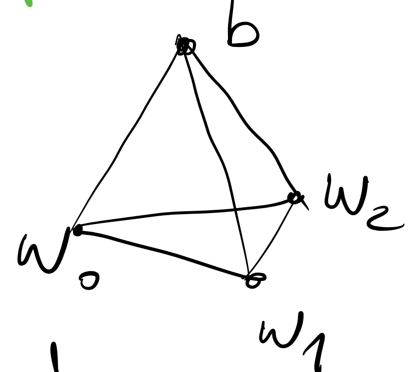
a subcomplex of $C_n(Y)$ that we denote by $LC_n(Y)$.

Each $b \in Y$ determines a homomorphism $b : LS_p(Y) \to LS_{p+1}(Y)$ defined by:

$$b([w_0, \dots, w_p]) = [b, w_0, \dots, w_p]$$

& extended to all of $LS_p(Y)$ linearly.

CONE OPERATOR



b sends a linear chain to the cone that has this chain as a base & whose tip is b

Let's compute

$$\partial\left(b[w_0,\dots,w_p]\right) = \partial\left([b,w_0,\dots,w_p]\right)$$

$$= (-1)^0[w_0,\dots,w_p] + (-1)^1[b,w_1,\dots,w_p]$$

$$+ (-1)^2[b,w_0,w_2,\dots,w_p] + \dots + (-1)^p[b,w_0,\dots,w_{p-1}]$$

$$= [w_0,\dots,w_p] - b\Big([w_1,\dots,w_p] + (-1)^1[w_0,w_2,\dots,w_p]$$

$$+ \dots + (-1)^p[w_0,\dots,w_{p-1}]\Big) =$$

$$= [w_0,\dots,w_p] - b\,\partial[w_0,\dots,w_p] =$$

$$= (id - b\circ\partial)[w_0,\dots,w_p]$$

$$\Longrightarrow \partial b = id - b\circ\partial$$

b is a **CHAIN HOMOTOPY** between 0 and the identity on the augmented chain complex $LC_n(X)$.

Now we define a **SUBDIVISION HOMOMORPHISM** $sd_p : LS_p(Y) \to LS_p(Y)$ by induction on p.

$p = -1$

$$sd_{-1}([\emptyset]) = [\emptyset]$$

$$sd_{-1} = id$$

$p \geq 0$ for generators $\sigma \in LS_p(Y)$

$$sd_p(\sigma) = b_\sigma(sd_{p-1}(\partial\sigma)),$$

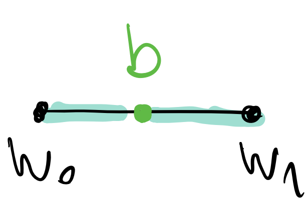
where b_σ is the barycenter of σ

$p = 0$

$$sd_0([w_0]) = w_0([\emptyset]) = [w_0]$$

$$\Rightarrow sd_0 = id$$

$p = 1$



$$sd_1([w_0, w_1]) =$$

$$= b\left((-1)^0[w_1] + (-1)^1[w_0]\right)$$

$$= b([w_1] - [w_0]) =$$

$= [b, w_1] - [b, w_0]$

sum of the 1-simplices in the barycentric subdivision with certain signs

(compare the def. of sd with that of the subdivision of a simplex)

sd is a chain map.

We prove this by induction:

$$\underset{\substack{\| \\ id}}{sd_{-1}} \circ \partial = \partial \circ \underset{\substack{\| \\ id}}{sd_0} \qquad \checkmark$$

$$\begin{tikzcd}
\cdots LS_{p+1}(Y) \arrow[r] \arrow[d, "sd_{p+1}"] & LS_p(Y) \arrow[r] \arrow[d, "sd_p"] & \cdots \arrow[r] & LS_0(Y) \arrow[r] \arrow[d, "id"] & LS_{-1}(Y) \arrow[r] \arrow[d, "id"] & 0 \\
\cdots LS_{p+1}(Y) \arrow[r] & LS_p(Y) \arrow[r] & \cdots \arrow[r] & LS_0(Y) \arrow[r] & LS_{-1}(Y) \arrow[r] & 0
\end{tikzcd}$$

Squares up to p commute.

$$\partial b_\sigma = id - b_\sigma \partial$$

induction step:

$$\partial(sd_{p+1}(\sigma)) = \partial(b_\sigma(sd_p(\partial\sigma))) =$$

$$\partial sd_p = sd_{p-1}\partial$$

$$= sd_p(\partial\sigma) - b_\sigma(\partial sd_p(\partial\sigma)) =$$

$$\partial\partial = 0$$

$$= sd_p(\partial\sigma) - b_\sigma(\cancelto{0}{sd_{p-1}\partial(\partial\sigma)}) =$$

$$= sd_p(\partial\sigma)$$

Next we build a chain homotopy $D: LS_p(Y) \to LS_{p+1}(Y)$ between sd and id.

chain map

$$D_p : LS_p(Y) \to LS_{p+1}(Y)$$

s.t. $\partial D + D\partial = id - sd$

We define D inductively.

$$
\begin{tikzcd}
\cdots LS_{p+1}(Y) \arrow[r] & LS_p(Y) \arrow[r] & \cdots \arrow[r] & LS_0(Y) \arrow[r] \arrow[d, "sd"', "id"] & LS_{-1}(Y) \arrow[r] \arrow[d, "sd"', "id"] \arrow[dl, "D_{-1}"'] & 0 \arrow[dl, "D_{-2}"] \\
\cdots LS_{p+1}(Y) \arrow[r] & LS_p(Y) \arrow[r] & \cdots \arrow[r] & LS_0(Y) \arrow[r] & LS_{-1}(Y) \arrow[r] & 0
\end{tikzcd}
$$

$D_{-2} = 0$

$D_{-1} = 0$

$$\partial D_{-1} + D_2 \partial = 0 + 0 = 0$$

$$id - sd_{-1} = id - id = 0$$

$$\Rightarrow \partial D_{-1} + D_{-2}\partial = id - sd_{-1}$$

$p \geq 0$ We define D_p inductively.

$\sigma \in LS_p(Y)$ a simplex

$$D_p(\sigma) = b_\sigma\left(\sigma - D_{p-1}(\partial\sigma)\right)$$

barycenter of σ

We check using induction that $\partial D + D\partial = id - sd$. Assume that all maps up to D_p satisfy this.

$$\partial D_{p+1}\sigma = \partial\big(b_\sigma(\sigma - D_p(\partial\sigma))\big) = \qquad \partial b_\sigma = id - b_\sigma\partial$$

$$= \sigma - D_p(\partial\sigma) - b_\sigma\big(\partial(\sigma - D_p(\partial\sigma))\big)$$

$$= \sigma - D_p(\partial\sigma) - b_\sigma(\partial\sigma - \partial D_p(\partial\sigma))$$

$$\begin{tikzcd}
\cdots \arrow[r] & LS_{p+2}(Y) \arrow[r] \arrow[d] & LS_{p+1}(Y) \arrow[r, "\partial"] \arrow[d, "sd" right, "id" left] \arrow[dl, "D_{p+1}"'] & LS_p(Y) \arrow[r] \arrow[d, "sd" right, "id" left] \arrow[dl, "D_p"'] & LS_{p-1}(Y) \arrow[r] \arrow[d] \arrow[dl, "D_{p-1}"'] & \cdots \\
\cdots \arrow[r] & LS_{p+2}(Y) \arrow[r, "\partial"'] & LS_{p+1}(Y) \arrow[r] & LS_p(Y) \arrow[r] & LS_{p-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

IH: $\partial D_p + D_{p-1}\partial = id - sd$

$$\Rightarrow id - \partial D_p = sd_p + D_{p-1}\partial$$

$$\overset{I.H}{=} \sigma - D_p(\partial\sigma) - b_\sigma\big(sd_p(\partial\sigma) + D_{p-1}\partial(\partial\sigma)\big)$$

$$= \sigma - D_p\partial(\sigma) - b_\sigma\, sd_p(\partial\sigma)$$

$$= \sigma - D_p\partial(\sigma) - sd_{p+1}(\sigma)$$

$$\Rightarrow \partial D_{p+1}(\sigma) + D_p\partial(\sigma) = \sigma - sd_{p+1}(\sigma)$$

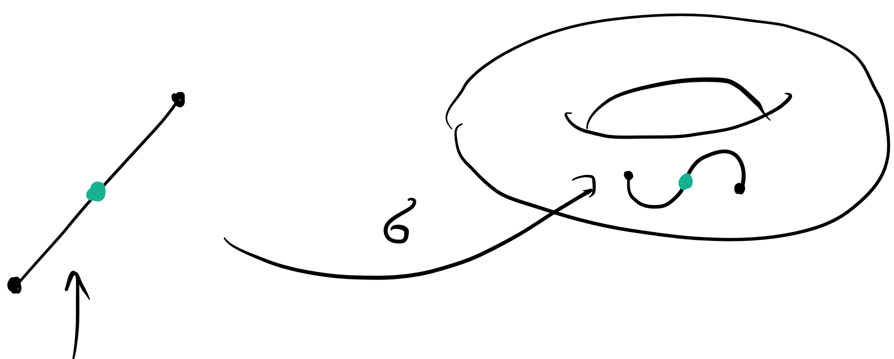
③ BARYCENTRIC SUBDIVISION OF GENERAL CHAINS

Let X be a topological space.

Define homomorphisms

$$sd = sd_p : S_p(X) \to S_p(X)$$

on generators $\sigma\ (\sigma : \Delta^p \to X)$



We subdivide this space, which is a convex subset in $\mathbb{R}^{p+1}$

$$sd(\sigma) = \sigma_c\big(sd(\underbrace{id : \Delta^p \to \Delta^p})\big)$$

$$(id : \Delta^p \to \Delta^p) \in LS_p(\Delta^p) \subset S_p(\Delta^n)$$

$\sigma : \Delta^p \to X$ induces $\sigma_c : S_p(\Delta^p) \to S_p(X)$.

$$\begin{tikzcd} \mathrm{id} \in S_p(\Delta^p) \arrow[r, "\sigma_c"] \arrow[d, "sd"] & S_p(X) \ni \sigma \arrow[d, "sd"] \\ S_p(\Delta^p) \ni sd(\mathrm{id}) \arrow[r, "\sigma_c"] & S_p(X) \end{tikzcd}$$

sd is a chain map

$$\partial(sd(\sigma)) = \partial\sigma_c(sd(\mathrm{id} : \Delta^p \to \Delta^p)) =$$

$$= \sigma_c \partial(sd(\mathrm{id} : \Delta^p \to \Delta^p)) =$$

σ_c is a chain map

$$= \sigma_c \, sd(\partial \, \mathrm{id}_{\Delta^p}) =$$

$$= \sigma_c \, sd\left(\sum_{i=0}^{p} (-1)^i \, \mathrm{id}_{\Delta_i^p}\right)$$

restriction of id to the ith face of Δ^p

$$= \sum_{i=0}^{p} (-1)^i \sigma_c \, sd(id_{\Delta_i^p})$$

sum of signed simplices in the barycentric subdivision of Δ_i^p

$$= \sum_{i=0}^{p} (-1)^i \, sd\left(\sigma_c|_{\Delta_i^p}\right)$$

$$= sd\left(\sum_{i=0}^{p} (-1)^i \sigma_c|_{\Delta_i^p}\right) =$$

$$= sd(\partial\sigma)$$

In a similar fashion we define

$D: S_p(X) \to S_{p+1}(X)$

$D(\sigma) = \sigma_c(D(id_{\Delta^p}))$

here we take the D defined for singular chains

D is a chain homotopy between sd & id.

σ_c is a chain map

$$\partial D(\sigma) = \partial(\sigma_c(D(id_{\Delta^p}))) =$$

$$= \sigma_c(\partial D(id_{\Delta^p})) =$$

D is a chain homotopy for linear chains

$$= \sigma_c\left(id_{\Delta^p} - sd(id_{\Delta^p}) - D\partial(id_{\Delta^p})\right)$$

$$\sigma_c D\partial(id_{\Delta^p}) = \sigma_c D\left(\sum_{i=0}^{p}(-1)^i id_{\Delta_i^p}\right) = \sum_{i=0}^{p}(-1)^i \sigma_c D id_{\Delta_i^p} = \sum_{i=0}^{p}(-1)^i D(\sigma_i) = D\sum_{i=0}^{p}(-1)^i \sigma_i = D\partial\sigma$$

$$= \sigma - sd(\sigma) - D\partial(\sigma) =$$

$$= (id - sd - D\partial)(\sigma)$$

Before we prove Theorem 1, let's recall

Lebesgue's number Lemma:

If the metric space (X, d) is compact & an open cover of X is given, then there exists a number $\delta > 0$ such that every subset of X having diameter less than δ is contained in some member of the cover.