PROOF OF THEOREM 1

Let $\mathcal{U}$ be a covering as in the statement of Theorem 1. Let $\sigma \in S_p(X)$ be a singular simplex.

Then $\{ \sigma^{-1}(\mathring{U}) \mid U \in \mathcal{U} \}$ is an open covering of Δ^p. Δ^p is compact, so we can select the Lebesgue number δ of this covering.

Pick $m \in \mathbb{N}$ large enough that

$$\left(\frac{p}{p+1}\right)^{m_\delta} \sqrt{2} \leq \delta.$$

diameter of an p-simplex

m will determine how much we have to subdivide simplices so that each lies in some $U \in \mathcal{U}$

If we use sd m-times on

σ we get a chain consisting of singular simplices, of which each lies in some $U \in \mathcal{U}$.

$$\sigma_c(sd^m(id_{\Delta^p})) = sd^m(\sigma) \in S_p^{\mathcal{U}}(X).$$

For each p-simplex σ we select the m_σ in a way that it is the smallest non-negative integer for which $sd^{m_\sigma}(\sigma) \in S_p^{\mathcal{U}}(X)$

$(m_\sigma = 0 \Leftrightarrow \sigma \in S_p^{\mathcal{U}}(X))$.

We define

$$\overline{D} : S_p(X) \to S_{p+1}(X)$$

$$\overline{D}(\sigma) = \sum_{j=0}^{m_\sigma - 1} D(sd^j(\sigma))$$

this is the D that we defined for singular chains

for σ a p-simplex

If $m_\sigma = 0$, $\overline{D}(\sigma) = 0$.

We calculate

$$(\partial\overline{D} + \overline{D}\partial)(\sigma) = \partial \sum_{j=0}^{m_\sigma - 1} D(sd^j(\sigma)) + \sum_{i=0}^{p} (-1)^i \,\overline{D}\,\sigma^i =$$

i-th face of σ

$$= \sum_{j=0}^{m_\sigma - 1} \partial D(sd^j(\sigma)) + \sum_{i=0}^{p} (-1)^i \sum_{j=0}^{m_{\sigma_i} - 1} D(sd^j(\sigma^i))$$

$$= \sum_{j=0}^{m_\sigma - 1} \left(sd^j(\sigma) - sd^{j+1}(\sigma) - D\partial(sd^j(\sigma)) \right)$$

$$+ \sum_{i=0}^{p} (-1)^i \sum_{j=0}^{m_{\sigma_i} - 1} D(sd^j(\sigma^i)) =$$

$$= \sigma - sd^{m_\sigma}(\sigma) - \sum_{j=0}^{m_\sigma - 1} D(sd^j(\partial\sigma)) +$$

$$+ \sum_{i=0}^{p} (-1)^i \sum_{j=0}^{m_{\sigma_i}-1} D\left(sd^j(\sigma^i)\right) =$$

$$= \sigma - sd^{m_\sigma}(\sigma) - \sum_{j=0}^{m_\sigma - 1} \sum_{i=0}^{p} (-1)^i D\left(sd^j(\sigma_i)\right)$$

$$+ \sum_{i=0}^{p} (-1)^i \sum_{j=0}^{m_{\sigma_i}-1} D\left(sd^j(\sigma^i)\right) =$$

$$= \sigma - sd^{m_\sigma}(\sigma) + \sum_{i=0}^{p} (-1)^i \sum_{j=m_{\sigma_i}}^{m_\sigma - 1} D\left(sd^j(\sigma^i)\right)$$

$$(m_{\sigma_i} \le m_\sigma)$$

We set

$$\rho(\sigma) := \sigma - \partial \bar{D}(\sigma) - \bar{D}\partial(\sigma)$$

Note that $\rho(\sigma) \in S_p^{\mathcal{U}}(X)$.

This ρ is a map: $S_p(X) \to S_p^{\mathcal{U}}(X)$.

ρ is a chain map:

$$\partial\rho(\sigma) = \partial\sigma - \partial\partial\bar{D}(\sigma) - \partial\bar{D}\partial(\sigma)$$

$$= \partial\sigma - \partial\bar{D}\partial(\sigma)$$

$$= \partial\sigma - \partial\bar{D}\partial(\sigma) - \bar{D}\partial\partial(\sigma)$$

$$= \rho(\partial\sigma)$$

$\Rightarrow \partial\bar{D} - \bar{D}\partial = id - i_c^u \circ \rho$,

where $i_c^u : C_n^u(X) \to C_n(X)$ is the inclusion.

$\bar{D}$ is a chain homotopy from $i_c^u \circ \rho$ to id.

Also, $\rho \circ i_c^u = \rho(i_c^u(\sigma)) =$

$$= \sigma - \overbrace{\partial\bar{D}(i_c^u(\sigma))}^{=0} - \overbrace{\bar{D}\partial(i_c^u(\sigma))}^{=0}$$

$= id$

so ρ is the chain homotopy inverse of $i_c^{\mathcal{U}}$.

It follows from homotopy invariance statements that $i_*^{\mathcal{U}}$ is an isomorphism $H_p^{\mathcal{U}}(X) \xrightarrow{i_*^{\mathcal{U}}} H_p(X)$.

PROOF OF EXCISION THEOREM

Let $\mathcal{U} = \{A, B\}$ such that $\mathring{A} \cup \mathring{B} = X$.

$$i_c^{\mathcal{U}} : C_n^{\mathcal{U}}(X) \to C_n(X)$$

is a chain equivalence. From proof of theorem 1 we get maps ρ & $\overline{D}$ that map simplices in A to simplices in A.

ρ and $\overline{D}$ induce maps on quotients

$$\rho : S_p(X) / S_p(A) \to S_p^{\mathcal{U}}(X) / S_p(A)$$

$$\overline{D} : S_p(X) / S_p(A) \to S_{p+1}(X) / S_{p+1}(A)$$

It still holds that

$$\partial \overline{D} + \overline{D} \partial = id - i_c^{\mathcal{U}} \circ \rho$$

and that

$$i_c^{\mathcal{U}} : C_n^{\mathcal{U}}(X) / C_n(A) \to C_n(X) / C_n(A)$$

is a chain equivalence and consequently it induces an isomorphism on homology.

The map

$$S_p(B) \big/ S_p(A \cap B) \longrightarrow S_p^{\mathcal{U}}(X) \big/ S_p(A)$$

induced by inclusion is an isomorphism since both quotient groups are free with the basis singular p-simplices in B that do not lie in A. $\Rightarrow$

$$H_p(X,A) \cong H_p\left(C_n^{\mathcal{U}}(X) \big/ C_n(A) \right)$$

$$\cong H_p(B, A \cap B).$$

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