Since $c$ is a cycle

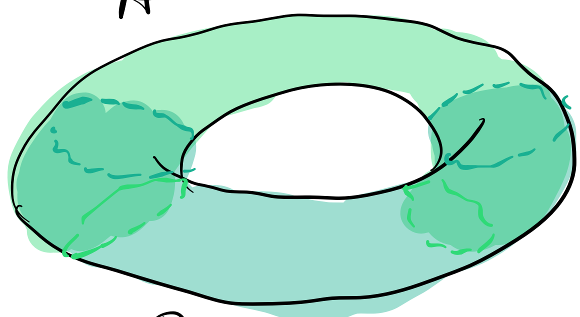
$0 = \partial c = \partial a + \partial b$ and therefore, $-\partial b = \partial a$.

$$\partial_* m = [\partial a] = [-\partial b] \in H_{p-1}(A \cap B).$$

## EXAMPLE

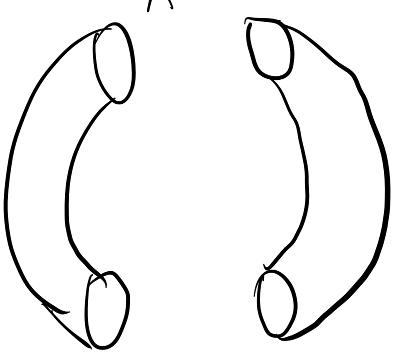
Compute homology groups of the torus using the Mayer-Vietoris sequence.

A



B

A



B

$A \cap B$ homotopy equivalent to $S^1 \sqcup S^1$

$A, B$ homotopy equivalent to $S^1$

The Mayer-Vietoris sequence is torus

$$\cdots H_p(A\cap B)\to H_p(A)\oplus H_p(B)\to H_p(T)\to H_{p-1}(A\cap B)\to\cdots$$

For $p>2$ we get

$$\cdots\to 0\oplus 0\to H_p(T)\to 0\to\cdots$$

$\tilde{H}_p(S^1)=0 \quad p\geq 2$

and therefore $H_p(T)=0$ for $p>2$.

For $p=0$ consider

$$\cancelto{0}{H_2(A)}\oplus\cancelto{0}{H_2(B)}\to H_2(T^2)\to H_1(A\cap B)\to H_1(A)\oplus H_1(B)$$

$$\to H_1(T^2)\to\tilde{H}_0(A\cap B)\to\cancelto{0}{\tilde{H}_0(\ )}\oplus\cancelto{0}{\tilde{H}_0(\ )}$$

$$\to\tilde{H}_0(T)\to 0$$

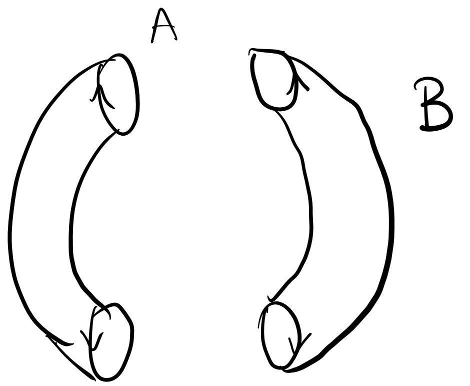
From here we get $\tilde{H}_0(T)=0$. Now consider

$$0\to H_2(T^2)\xrightarrow{\partial} H_1(A\cap B)\xrightarrow{(i_*^A,\,-i_*^B)} H_1(A)\oplus H_1(B)$$

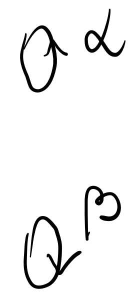
$$\to H_1(T^2)\xrightarrow{\partial_*} H_0(A\cap B)\to 0.$$

Let us first compute $H_2(T^2)$.
$\partial$ is injective, so $H_2(T^2)$ is isomorphic to $\mathrm{Im}\,\partial_*$. By exactness, $\mathrm{Im}\,\partial_* = \ker(i^A_*; -i^B_*)$. So the next step is to determine $\ker(i^A_*; -i^B_*)$.

For this we choose the cycles generating the homologies of $A$, $B$ & $A \cap B$.




$A \cap B$ homotopy equivalent to $S^1 \sqcup S^1$




Now $H_1(A \cap B) \cong \langle \alpha \rangle \oplus \langle \beta \rangle$.

In $H_1(A)$ & $H_1(B)$ $\alpha = \beta$, so

$$(i^A_*, -i^B_*)(\alpha, 0) = (i^A_*, -i^B_*)(0, \beta) = (\alpha, -\beta)$$

Hence, $(i_*^A, -i_*^B)$ can be represented by

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$

Hence, $H_2(T) = \operatorname{Im} \partial_* = \ker \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} =$

$$= \langle \alpha - \beta \rangle = \mathbb{Z}$$

For $H_1(T^2)$ we focus on the following piece of the MVS:

$$\cdots H_1(A \cap B) \xrightarrow{(i_*^A, -i_*^B)} H_1(A) \oplus H_1(B) \xrightarrow{j_A^* + j_B^*} H_1(T^2) \xrightarrow{\partial_*} H_0(A \cap B) \to \cdots$$

We know all the groups except $H_1(T^2)$. Making an argument similar to the previous one, we can show that

$(i_*^A - i_*^B) : H_0(A \cap B) \to H_0(A) \oplus H_0(B)$

can be represented by $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$.

Now we produce a SES from the sequence above:

$$0 \to \ker \partial_* \to H_1(T^2) \to \operatorname{Im} \partial_* \to 0$$

$$\operatorname{Im} \partial_* = \ker\left(\left(i^A_*, - i^B_*\right)\right) = \ker \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \mathbb{Z}$$

We also know that

second iso theorem

$$\ker \partial_* = \operatorname{Im}\left(j_A^* + j_B^*\right) \cong \mathbb{Z} \oplus \mathbb{Z} \big/ \ker\left(j_A^* + j_B^*\right)$$

$$= \mathbb{Z} \oplus \mathbb{Z} \big/ \operatorname{Im}\left(i^A_*, - i^B_*\right)$$

$$\cong \mathbb{Z}$$

Hence, we have the following split SES ($\mathbb{Z}$ is free, so we have a right inverse)

$$0 \to \mathbb{Z} \to H_1(T) \to \mathbb{Z} \to 0$$

Thus $H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$.

So, for the torus, the homology groups are

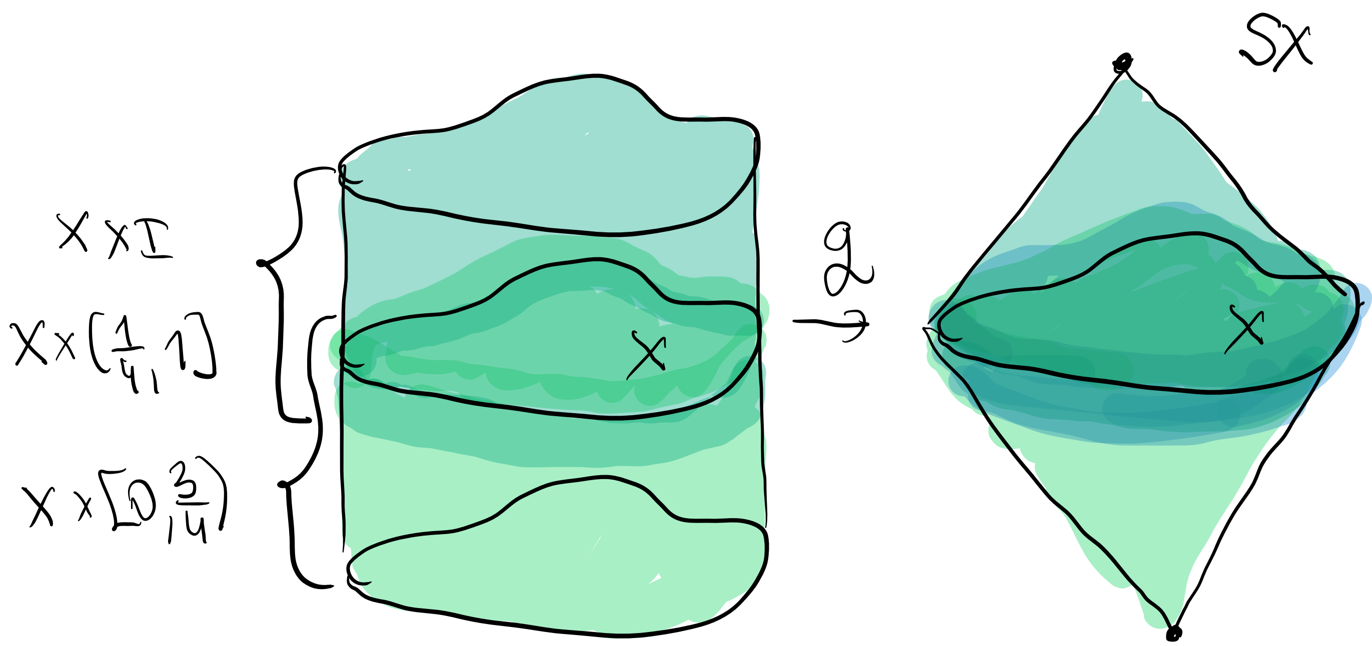
$$H_i(T^2) = \begin{cases} 0 & i \geq 2 \\ \mathbb{Z} & i = 2 \\ \mathbb{Z} \oplus \mathbb{Z} & i = 1 \\ \mathbb{Z} & i = 0 \end{cases}$$

EXAMPLES

Show that $\tilde{H}_p(X) \cong \tilde{H}_{p+1}(SX)$ for all $p$, where $SX$ is the suspension of $X$.

More generally, thinking of $SX$ as the union of two cones $CX$ with their base identified, compute the reduced homology groups of the union of any finite number of cones $CX$ with their bases identified. $(n=3)$.

Recall that the suspension of $X$, $SX$, is $X \times I$ with $X \times \{0\}$ and $X \times \{1\}$ collapsed into a point.




Let $A = g\left(X \times (\tfrac{1}{4}, 1]\right)$
$B = g\left(X \times [0, \tfrac{3}{4})\right)$.

The interiors of $A$ & $B$ cover $SX$.

$A$ & $B$ are both contractible and therefore have vanishing reduced homology

$A \cap B$ is homotopy equivalent to $X$.

Then the reduced Mayer-Vietoris sequence yields

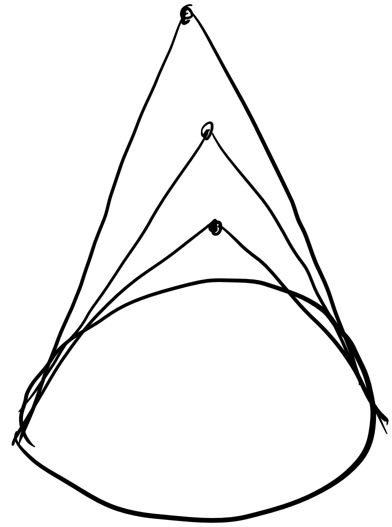
$$\to \tilde{H}_p(A) \oplus \tilde{H}_p(B) \to \tilde{H}_p(SX) \to \tilde{H}_{p-1}(X) \to \tilde{H}_{p-1}(A) \oplus \tilde{H}_{p-1}(B) \to$$

$0$ $0$

and hence $\tilde{H}_p(SX) \cong \tilde{H}_{p-1}(X)$.

Now consider $n=3$

We denote this space by $S_3X$.



We use the Mayer-Vietoris sequence for

$$S_3X = SX \cup_X CX :$$ both slisHley thickened

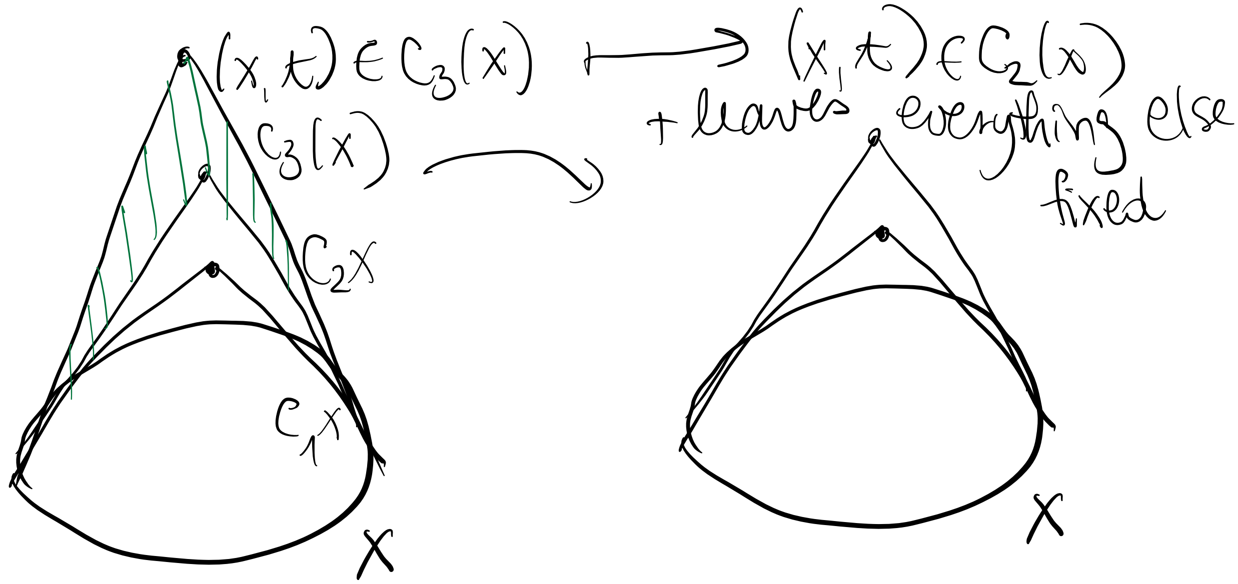
$$\dots \to \tilde{H}_p(X) \to \tilde{H}_p(SX) \oplus \tilde{H}_p(CX) \to \tilde{H}_p(S_3X) \to \dots$$

We have $\tilde{H}_p(CX) = 0$ for all $p$.

Furthermore, the morphism induced by inclusion $\tilde{H}_p(X) \to \tilde{H}_p(SX)$ is trivial since any cycle in $X$ is a boundary inside $SX$ (boundary of the cone, for example). Hence, this sequence simplifies to

$$0 \to \tilde{H}_p(SX) \to \tilde{H}_p(S_3 X) \to \tilde{H}_{p-1}(X) \to 0 \quad \forall p$$

Now observe that there exists a retraction $r: S_3 X \to SX$.




this retraction induces a map $r_*: \tilde{H}_p(S_3 X) \to \tilde{H}_p(SX)$ with

$r_* \circ i_* = id_{\tilde{H}_p(S_p)}$, where $i: SX \hookrightarrow S_3 X$

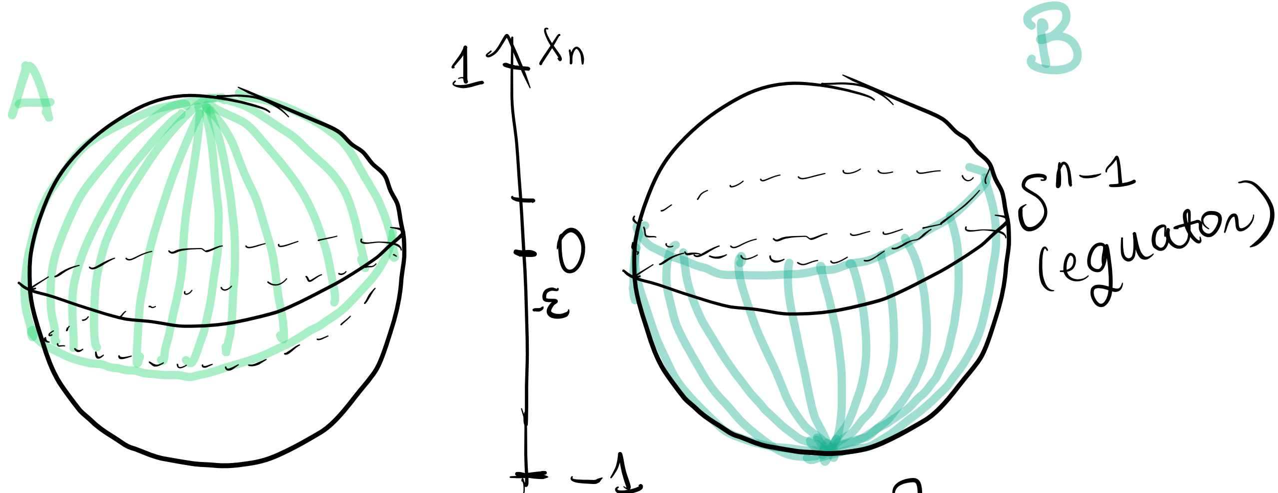
is an inclusion. This means that this SES is split. In particular,

$$\tilde{H}_p(S_3 X) \cong \tilde{H}_{p-1}(X) \oplus \tilde{H}_p(SX)$$

EXAMPLE (special case of the previous problem)

$X = S^n$, $n \geq 1$. $\quad \cong \tilde{H}_{p-1}(X) \oplus \tilde{H}_{p-1}(X)$

$S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1}, \; x_0^2 + \dots + x_n^2 = 1\}$




$A = \{(x_0, \dots, x_n) \in S^n : -\varepsilon < x_n \leq 1\}$

$B = \{(x_0, \dots, x_n) \in S^n : -1 \leq x_n < \varepsilon\}$

$A, B \cong (\mathring{B}^n)$ & so $A, B$ are contractible

$\Rightarrow \tilde{H}_p(A) = \tilde{H}_p(B) = 0 \quad \forall p.$

Also, $A \cap B \simeq S^{n-1}$ ($A \cap B \neq \emptyset$ since $n \geq 1$).

We now use the Mayer-Vietoris LES:

$$\to \tilde{H}_p(S^{n-1}) \to 0 \oplus 0 \to \tilde{H}_p(S^n) \to \tilde{H}_{p-1}(S^{n-1}) \to 0 \oplus 0 \to$$

$$\Rightarrow \tilde{H}_p(S^n) \cong \tilde{H}_{p-1}(S^{n-1}).$$

$$\Rightarrow \tilde{H}_p(S^n) \cong \cdots \cong \tilde{H}_{p-n}(S^0) = \begin{cases} \mathbb{Z} & p = n \\ 0 & p \neq n \end{cases}$$

There is also a version of Mayer-Vietoris for closed subsets

## MAYER-VIETORIS SEQUENCE FOR CLOSED SUBSETS (MV#2)

Let $X$ be a space & $A, B$ closed subsets of $X$ s.t. $X = A \cup B$. Also assume that $A$ is a strong deformation retract of its neighborhood in $X$ $U$, $B$ is a strong deformation retract of its neighborhood $V$ and $A \cap B$ is a strong deformation retract of $U \cap V$. Then