DEGREE OF MAPS $f: S^n \to S^n$

Let $f: S^n \to S^n$ be a map. Then f induces $f_*: \tilde{H}_n(S^n) \to \tilde{H}_n(S^n)$.
Since $\tilde{H}_n(S^n) \cong \mathbb{Z}$, there exists precisely one $d \in \mathbb{Z}$, such that $f_*(a) = da \quad \forall a \in \mathbb{Z}$.

This number d is called the **DEGREE of f** and is denoted $\deg(f) \in \mathbb{Z}$.

SIMPLE PROPERTIES OF DEGREE

1. $\deg(id) = 1$
2. $S^n \xrightarrow{f} S^n \xrightarrow{g} S^n \Rightarrow \deg(g \circ f) = \deg g \cdot \deg f$
3. If $f \simeq g : S^n \to S^n \Rightarrow \deg(f) = \deg(g)$.

Proof

1. follows since $(id)_* = id$.

② follows since $(g \circ f)_* = g_* \circ f_*$

③ If $f \simeq g$, then $f_* = g_*$, so $\deg(f) = \deg(g)$.

PROPOSITION

Let $S^n \subset \mathbb{R}^{n+1}$ be the n-dim sphere, write the elements of S^n as $(x_0, \dots, x_n)$. Let $f : S^n \to S^n$ be the map

$$f(x_0, \dots, x_n) := (-x_0, x_1, \dots, x_n).$$

Then $\deg(f) = -1$.

Proof

Let $n = 0$. Then $f : \{-1, 1\} \to \{-1, 1\}$ is the map $f(-1) = 1$, $f(1) = -1$.

$$\begin{tikzcd}
H_0(\{-1\}) \oplus H_0(\{1\}) \arrow[r, "\overset{2}{\cong}"] & H_0(S^0) \arrow[d, "\varepsilon_*"] \\
\mathbb{Z} \oplus \mathbb{Z} & \\
(a, b) \arrow[r, mapsto] & a + b \in \mathbb{Z}
\end{tikzcd}$$

the kernel of the above map

$$\begin{tikzcd}
\tilde{H}_0(S^0) \arrow[d, "f_*"] & \{(a,-a),\, a\in\mathbb{Z}\} \subset \mathbb{Z}\oplus\mathbb{Z} \arrow[l, "g"', "\cong"] \arrow[d, "(a,-a)\mapsto(-a,a)"] \\
\tilde{H}_0(S^0) \arrow[r, "g^{-1}"'] & \{(a,-a),\, a\in\mathbb{Z}\} \subset \mathbb{Z}\oplus\mathbb{Z}
\end{tikzcd}$$

$\cong \mathbb{Z}$ $\qquad$ $\cong \mathbb{Z}$

From this it immediately follows that

$\deg(f) = -1$.

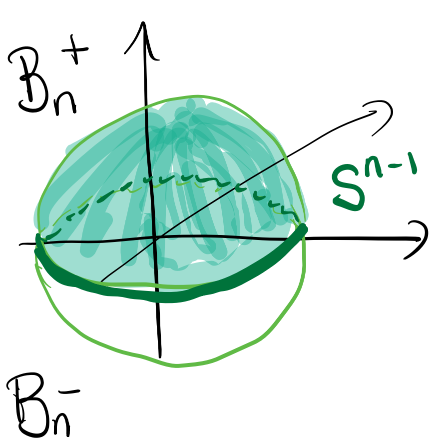
Now we proceed by induction.

Let $n \geq 1$. Suppose that the statement is true for S^k $\forall\ 0 \leq k < n$.

We now prove it for n.

Consider

$$B_+^n = \{(x_0, \dots, x_n) \in S^n : x_n \geq 0\}$$
$$B_-^n = \{(x_0, \dots, x_n) \in S^n : x_n \leq 0\}$$



We have

$$f(B_\pm^n) = B_\pm^n$$
$$f(S^{n-1}) = S^{n-1}$$

$$\begin{tikzcd}
\tilde{H}_n(S^n) \arrow[r, "\cong", "\text{good pair}"'] \arrow[d, "f_*"] & H_n(S^n, B^n_+) \arrow[d, "f_*"] & H_n(B^n_-, S^{n-1}) \arrow[l, "\cong"', "\text{excision + homology}"] \arrow[r, "\cong", "\text{LES of a pair}"'] \arrow[d, "f_*"] & \tilde{H}_{n-1}(S^{n-1}) \arrow[d, "(f|_{S^{n-1}})_*"] \\
\tilde{H}_n(S^n) \arrow[r, "\cong"] & H_n(S^n, B^n_+) & H_n(B^n_-, S^{n-1}) \arrow[l, "\cong"'] \arrow[r, "\cong"] & \tilde{H}_{n-1}(S^{n-1})
\end{tikzcd}$$

By induction $\deg(f|_{S^{n-1}}) = -1 \Rightarrow$ all vertical maps are multiplications by -1. $\blacksquare$

COROLLARY

Let $0 \leq i \leq n$, $\tau_i : S^n \to S^n$,

$$\tau_i(x_0, \ldots, x_n) := (x_0, \ldots, -x_i, \ldots, x_n).$$

Then $\deg(\tau_i) = -1$.

Proof

Show that $\tau_i \simeq \tau_{i-1} \simeq \ldots \simeq \tau_0$. (exercise)

$\Rightarrow \deg \tau_i = \deg \tau_0$

Hint for the homotopy: 2D case

$$\underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}_{\text{rotation in the 2-D plane by 180 deg.}} \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{\tau_0} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\tau_1}$$

by angles

$$H((x_0,x_1),t) = \begin{bmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{bmatrix} \begin{bmatrix} -x_0 \\ x_1 \end{bmatrix} \qquad t \in [0,1]$$

$$H((x_0,x_1),0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -x_0 \\ x_1 \end{bmatrix} = (-x_0, x_1) = \tau_0(x_0,x_1)$$

$$H((x_0,x_1),1) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -x_0 \\ x_1 \end{bmatrix} = (x_0, -x_1) = \tau_1(x_0,x_1)$$

IMPORTANT EXAMPLE

(the antipodal map)

Let $\sigma : S^n \to S^n$ be the map

$$\sigma(x) := -x.$$

Then $\deg \sigma = (-1)^{n+1}$.

Proof

$\sigma = \tau_0 \circ \tau_1 \circ \dots \circ \tau_n$

$\Rightarrow \deg \sigma = \deg \tau_0 \cdot \deg \tau_1 \cdots \deg \tau_n$

$= (-1)^{n+1}$

$\blacksquare$

COROLLARY

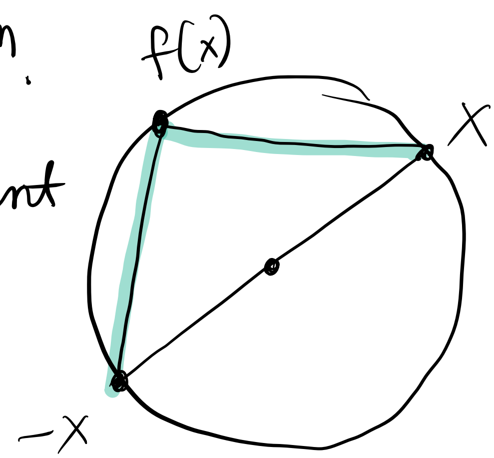
If $n =$ even $\Rightarrow \sigma \not\simeq id$.

COROLLARY

Let n be even and $f: S^n \to S^n$. Then there exists $x \in S^n$, s.t. $f(x) = \pm x$.

Proof

Suppose by contradiction that $f(x) \neq x$, $f(x) \neq -x$ $\forall x \in S^n$.



the straight segment in B^{n+1} connecting x to $f(x)$ does not pass through 0.

The same also holds for the segment connecting $-x$ to $f(x)$.

Consider $F: S^n \times I \to S^n$

$G: S^n \times I \to S^n$:

the denominators never vanish $\forall x \in S^n$, $t \in [0,1]$

$$F(x,t) := \frac{t f(x) + (1-t)x}{\| t f(x) + (1-t)x \|}$$

$$G(x,t) := \frac{t \cdot (-x) + (1-t) f(x)}{\| t \cdot (-x) + (1-t) f(x) \|}$$

F is a homotopy between id & f.
G is a homotopy between f & the antipodal map.

$\Rightarrow \deg(f) = 1$ & $\deg(f) = (-1)^{n+1} = -1$

n is even

Contradiction.

$\blacksquare$