Define a homotopy $F: S^n \times I \to S^n$ as follows:

$$F(x,t) := \begin{cases} f(x) & 2\varepsilon \leq |x| \\ f(x) - t\left(2 - \frac{|x|}{\varepsilon}\right) g(x) & \varepsilon \leq |x| \leq 2\varepsilon \\ f(x) - t g(x) & |x| \leq \varepsilon \end{cases}$$

interpolation

F is well-defined and continuous.

$F(x,0) = f(x)$. Put $f_1(x) := F(x,1)$.

Note that $f_1(x) = x$ for all $|x| \leq \varepsilon$.

Claim: $\forall x \neq 0,\ f_1(x) \neq 0$.

Proof of claim: For $|x| \geq 2\varepsilon$, $f_1(x) = f(x)$.

assumption: f admits g only once at p

If $\varepsilon \leq |x| \leq 2\varepsilon$, then

homotopy formula

$$f_1(x) = 0 \iff f(x) = \frac{2\varepsilon - |x|}{\varepsilon} g(x)$$

But if the latter equality holds for some x, then

$$\frac{|g(x)|}{|f(x)|} = \frac{\varepsilon}{2\varepsilon - |x|} = \frac{1}{2 - \underbrace{\frac{|x|}{\varepsilon}}_{\text{this is between } 1 \,\&\, 2}} \geq \frac{1}{2}.$$

This is a contradiction with $\frac{|g(x)|}{|f(x)|} < \frac{1}{100}$.

If $|x| \leq \varepsilon$, then $f_1(x) = x$ and in this case the claim is obvious. $\blacksquare$

Claim: For $r > 0$ small enough we have: $\forall\ |x| \leq r,\ f_1^{-1}(x) = \{x\}$.

Proof. If the claim doesn't hold, then $\exists\ r_n \to 0$ and points $|x_n| \leq r_n$ and points y_n with $\underline{|y_n| > \varepsilon}$ s.t. $f_1(y_n) = x_n$.

If $|y_n| \leq \varepsilon$, $f_1(y_n) = y_n$

Since S^n is compact there exists

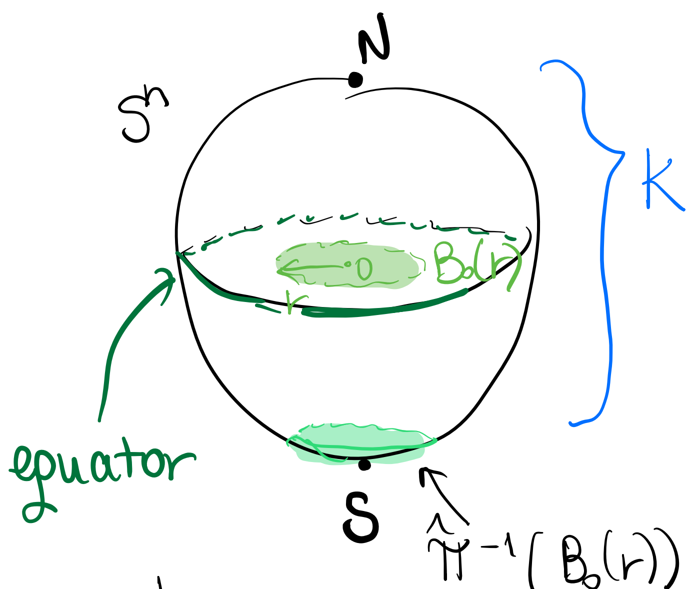
a subsequence of y_n, y_{n_k} that converges to $y := \lim_{k\to\infty} y_{n_k} \in S^n$. By continuity

$f_1(y) = 0$ because $f_1(y_{n_k}) = x_{n_k} \longrightarrow 0$.

But $y \neq 0$. Contradiction. ∎

It follows from the previous claim that

$$f_1(\underbrace{S^n \setminus B_0(r)}_{K}) \subset \underbrace{S^n \setminus B_0(r)}_{K}$$



$$f_1\big|_{\partial K} = id_{\partial K}$$

$f_1(B_0(r)) = B_0(r)$, in fact $f_1\big|_{B_0(r)} = id_{B_0(r)}$.

$$f_1\big|_K : (K, \partial K) \longrightarrow (K, \partial K)$$

Claim: $f_1\big|_K$ is homotopic to id_K, rel ∂K.

Proof Identify $K = B(R)$. Do a stereographical projection from $\boxed{\text{nbhd of } N}$ the south pole and identify K with the image. The image is a ball and hence convex.

Therefore we can take the standard linear homotopy $G(x,t) = tx + (1-t) f_1(x)$ to homotope f_1 to the identity.

CONCLUSION: $f \simeq f_1$ and $f_1 \simeq id$

$$\Rightarrow \deg f = \deg id = +1 = \varepsilon_0(f)$$

Step 2

Assume $Df_{(0)}$ general.

Put $A = Df_{(0)}$. Since 0 is a regular value, then A when viewed as a $n \times n$ matrix is non-singular.

Consider $h := \widehat{A}^{-1} \circ f$. Observe that

$Dh_{(0)} = id \Rightarrow$ by step 1 we obtain:

$$\underset{\substack{\uparrow \\ \text{step 1}}}{1 =} \deg h = \deg \hat{A}^{-1} \cdot \deg f =$$

$$= (\varepsilon_p(f))^{-1} \cdot \deg f$$

$\Rightarrow \deg f = \varepsilon_p(f)$ ∎

We want to generalize this further to allow non-singleton preimages.

Consider the following. Let Y be a space with a base point $y_0 \in Y$. Let $E_1, \dots, E_k$ be disjoint open subsets of S^n s.t. $E_i \approx \mathbb{R}^n$ $\forall i$.

Let $f : S^n \to Y$ be a map

$$f(S^n \setminus (E_1 \cup \dots \cup E_k)) = \{y_0\}.$$

Note that $S^n / S^n \setminus E_i \approx \hat{E}_i \approx \mathbb{R}^n \cup \{\infty\} \approx S^n$ for all i.

$\uparrow$ one-point compactification of E_i

$\Rightarrow S^n / S^n - (E_1 \cup \dots \cup E_k) \approx S_1^n \vee \dots \vee S_k^n$, where

$S_j^n := S^n / S^n \setminus E_j$.

Clearly, f factors as a composition as $f = h \circ g$, $S^n \xrightarrow{g} S_1^n \vee \dots \vee S_k^n \xrightarrow{h} Y$.

Put $i_j : S_j^n \longrightarrow S_1^n \vee \dots \vee S_k^n$ to be the inclusion, $p_j : S_1^n \vee \dots \vee S_k^n \to S_j^n$ is the projection. Then

$\bigoplus_{j=1}^{k} \tilde{H}_p(S_j^n) \xrightarrow{\cong} \tilde{H}_p(S_1^n \vee S_2^n \dots \vee S_k^n)$ and the isomorphism is induced by $\bigoplus_{j=1}^{k} (i_j)_*$.

(Proposition about wedge product from class)

The inverse of this map is $\bigoplus_{j=1}^{k} (p_j)_*$.

$$\sum_{j=1}^{k} (i_j)_* \circ (p_j)_* = id_{\tilde{H}_n(S_1^n \vee \dots \vee S_k^n)}$$

Define g_j & h_j as follows.

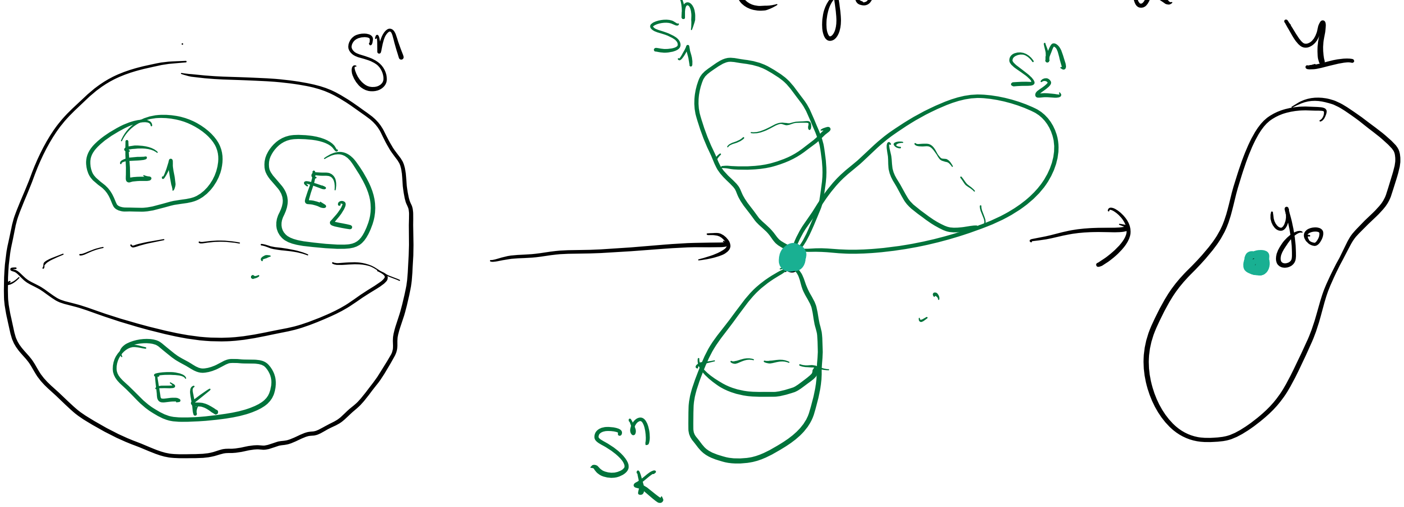
$$\begin{tikzcd}
S^n \arrow[r, "g"] \arrow[dr, "g_j"'] & S_1^n \vee \dots \vee S_k^n \arrow[r, "h"] \arrow[d, shift right, "p_j"'] & Y \\
& S_j^n \arrow[u, shift right, "i_j"'] \arrow[ur, "h_j"'] &
\end{tikzcd}$$

We also define $f_j : S^n \to Y$

$$f_j := h_j \circ g_j$$

collapses all E_i except E_j and the complements to a point, then applies f & finally push this to Y

More precisely, $f_j(x) = \begin{cases} f(x) & x \in E_j \\ y_0 & x \notin E_j \end{cases}$



THEOREM

$$f_* = \sum_{j=1}^{k} (f_j)_* : \tilde{H}_n(S^n) \to \tilde{H}_n(Y).$$

Proof Let $\alpha \in \tilde{H}_n(S^n)$.

$$g_*(\alpha) = \sum_{j=1}^{k} (i_j)_* (p_j)_* g_*(\alpha) =$$

$$= \sum_{j=1}^{k} (i_j)_* (g_j)_* (\alpha)$$

$$\Longrightarrow f_*(\alpha) = h_* g_*(\alpha) =$$

$$= \sum_{j=1}^{k} h_* (i_j)_* (g_j)_* (\alpha) =$$

$$= \sum_{j=1}^{k} (h_j)_* (g_j)_* (\alpha) =$$

$$= \sum_{j=1}^{k} (f_j)_* (\alpha)$$

$\blacksquare$

COROLLARY

Let $f: S^n \to S^n$ be a smooth map and let $p \in S^n$ be a regular value. Assume that $f^{-1}(p) = \{a_1, \dots, a_k\}$.

Then
$$\deg f = \sum_{j=1}^{k} \varepsilon_{a_j}(f).$$

↑ local degree of f at a_j

If $f^{-1}(p) = \phi$ (i.e. f is not surjective), then $\deg f = 0$.

(Note: this result is independent of homology theory as long as the coefficient group is $\mathbb{Z}$)

Proof

Assume first that $f^{-1}(p) \neq \phi$. By the implicit function theorem there exists an open ball $B \subset S^n$ around p s.t.
$f^{-1}(B) = \bigsqcup_{j=1}^{k} B_j$, where B_j is an open