Consider the LES of $(K^{(n)}, K^{(n-1)})$:

$$\to H_{p+1}(K^{(n)}, K^{(n-1)}) \xrightarrow{\partial_*} H_p(K^{(n-1)}) \xrightarrow{i_*} H_p(K^{(n)}) \to H_p(K^{(n)}, K^{(n-1)}) \xrightarrow{\partial_*}$$

- If $p \neq n$, then i_* is surjective.
- if $p \neq n-1$, then i_* is injective
- if $p \neq n, n-1$, then i_* is an isomorphism

Fix a positive index $p > 0$. By induction on n it is easy to show that $H_p(K^{(n)}) = 0 \ \forall p > n$.

Basis: $n = 0$, $H_p(K^{(0)}) = 0 \quad \forall j > 0$.

Similarly, it is easy to show that

for $p = n$ we have the exact sequence

$$(*)$$

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
0 \arrow[r] & H_n(K^{(n)}) \arrow[r, "j_n"] & H_n(K^{(n)}, K^{(n-1)}) \arrow[r, "\partial_n"] \arrow[dr, "\beta_n := j_{n-1} \circ \partial_n"'] & H_{n-1}(K^{(n-1)}) \arrow[r] \arrow[d, "j_{n-1}"] & H_{n-1}(K^{(n)}) \arrow[r] & 0 \\
 & & & H_{n-1}(K^{(n-1)}, K^{(n-2)}) \arrow[d, "\partial_{n-1}"] & & \\
 & & & H_{n-2}(K^{(n-2)}) & &
\end{tikzcd}$$

For $n+1$ we get (**)

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
0 \arrow[r] & H_{n+1}(K^{(n+1)}) \arrow[r, "j_{n+1}"] & H_{n+1}(K^{(n+1)},K^{(n)}) \arrow[r, "\partial_{n+1}"] \arrow[dr, "\beta_{n+1}"'] & H_n(K^{(n)}) \arrow[r, "i_n"] \arrow[d, "j_n"] & H_n(K^{(n+1)}) \arrow[r] & 0 \\
 & & & H_n(K^{(n)},K^{(n-1)}) \arrow[d, "\partial_n"] & & \\
 & & & H_{n-1}(K^{(n-1)}) & &
\end{tikzcd}$$

CLAIM $\beta_n \circ \beta_{n+1} = 0$.

Proof $\beta_n \circ \beta_{n+1} = j_{n-1} \circ \underbrace{\partial_n \circ j_n}_{0 \text{ since the above sequence is exact}} \circ \partial_{n+1} = 0$.

∎

CLAIM $\ker \beta_n = \ker \partial_n = \operatorname{Im} j_n$

$\operatorname{Im} \beta_{n+1} = j_n(\operatorname{Im} \partial_{n+1})$.

Proof

j_{n-1} is injective, $\ker \beta_n = \ker \partial_n$.

$\ker \partial_n = \operatorname{Im} j_n$ because the sequence $(*)$ is exact.

Second equality: exercise (diagram chasing).

Conclusion

$$\left.\begin{array}{ccc} H_n(K^{(n)}) & \xrightarrow[j_n]{\cong} & \ker \beta_n \\ \cup & & \\ \operatorname{Im}(\partial_{n+1}) & \xrightarrow[\cong]{j_n} & \operatorname{Im} \beta_{n+1} \end{array}\right\}$$

$$\Rightarrow H_n(K^{(n+1)}) \cong \operatorname{coker}(\partial_{n+1}) \xrightarrow[j_n]{\cong} \frac{\ker \beta_n}{\operatorname{Im} \beta_{n+1}}$$

Proof

In $(**)$ i_n is a surjection

$$H_n(K^{(n+1)}) \cong \frac{H_n(K^{(n)})}{\ker i_n} \cong \frac{H_n(K^{(n)})}{\operatorname{Im} \partial_{n+1}}$$

$$\cong \operatorname{coker}(\partial_{n+1})$$

CLAIM In

$$H_n(K^{(n+1)}) \xrightarrow{i_*} H_n(K^{(n+2)}) \xrightarrow{i_*} H_n(K^{(n+3)}) \to \cdots$$

all maps are isomorphisms.

Proof

Let $i \geq n$. then

$$\underbrace{H_{n+1}(K^{(i+1)}, K^{(i)})}_{\substack{\shortparallel \\ 0}} \to H_n(K^{(i)}) \to H_n(K^{(i+1)}) \to \underbrace{H_n(K^{(i+1)}, K^{(i)})}_{\substack{\shortparallel \\ 0}}$$

$\blacksquare$

COROLLARY

If K is finite dimensional ($\exists n_0$ s.t. $K^{(n_0)} = K^{(n_0+1)} = \cdots = K^{(n_0+i)} = \cdots = K$), then

$$H_n(K) \cong H_n(K^{(n+1)}) = \ker \partial_n / \operatorname{Im} \partial_{n+1}$$

$$H_n(K^{(n+1)}) \simeq H_n(K^{(n+2)}) \cdots H_n(K^{(n_0)}) = H_n(K)$$

DEFINITION

Define a chain complex

$$\cdots \to H_{n+1}(K^{(n+1)}, K^{(n)}) \xrightarrow{\beta_{n+1}} H_n(K^{(n)}, K^{(n-1)}) \xrightarrow{\beta_n} H_{n-1}(K^{(n-1)}, K^{(n-2)}) \to$$

$$\cdots \xrightarrow{\beta_2} H_1(K^{(1)}, K^{(0)}) \xrightarrow{\beta_1} H_0(K^{(0)}) \to 0$$

In this sequence β_1 is the connecting homomorphism from LES of $(K^{(1)}, K^{(0)})$.

We denote the n-homology group of this complex by $H_n^{CW}(K)$.

COROLLARY

If K is finite dimensional, then

$$H_n^{CW}(K) \cong H_n(K).$$

Remark: This statement holds for any CW-complex K, but we will not prove it here.

Since $H_p(K^{(p)}, K^{(p-1)})$ is isomorphic to $\bigoplus_{\sigma \in I_p} \mathbb{Z} \cdot \sigma$, we will actually want to work with a chain complex with chain groups

$$C_p^{CW}(K) := \bigoplus_{\sigma \in I_p} \mathbb{Z} \cdot \sigma .$$

DEFINITION

Define two homomorphisms:

$$C_n^{CW}(K) \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} H_n(K^{(n)}, K^{(n-1)})$$

Recall that $(B_\sigma^n, \partial B_\sigma^n) \xrightarrow{f_\sigma} (K^{(n)}, K^{(n-1)})$.

Denote by $[B_\sigma^n]$ the generator of $H_n(B_\sigma^n, \partial B_\sigma^n)$.

$$\Psi\left(\sum_\sigma n_\sigma \cdot \sigma\right) = \sum_\sigma n_\sigma (f_\sigma)_*([B_\sigma^n])$$

To define Φ, let $\phi_n : H_n(S^n, *) \to \mathbb{Z}$

be the unique homomorphism s.t.

$\phi_n([S^n]) = 1$.

Recall

bouquet of spheres

$$\begin{tikzcd}
(K^{(n)}, K^{(n-1)}) \arrow[r] \arrow[dr, "p_\sigma"'] & \left(K^{(n)}/K^{(n-1)}, *\right) \arrow[d] \\
& (S^n_\sigma, *)
\end{tikzcd}$$

Define

$$\Phi(\alpha) = \sum_{\sigma \in I_n} \phi_n((p_\sigma)_*(\alpha))\,\sigma.$$

We have already proved that Ψ is an isomorphism (the iso from the last lecture) - check the details yourself.

CLAIM

$\Phi = \Psi^{-1}$.

Proof

It is enough to prove that $\Phi \circ \Psi = id$ since Ψ is an isomorphism.

Since $C_n^{cw}(X)$ is generated by the n-cells σ, it is enough to check $\Phi \circ \Psi(\sigma) = \sigma \quad \forall \sigma$.

$$\Phi\Psi(\sigma) = \Phi\left((f_\sigma)_*[B^n_\sigma]\right) =$$

$$= \sum_\tau \phi_n\left((p_\tau)_*(f_\sigma)_*[B^n_\sigma]\right)\tau \overset{*}{=}$$

Note that $p_\tau \circ f_\sigma = \begin{cases} \text{const. at } * & \tau \neq \sigma \\ id & \tau = \sigma \end{cases}$

So

$$(*) = \phi_n([S^n]) \cdot \sigma = \sigma.$$

$\blacksquare$

Define a boundary operator $C^{cw}_{n+1}(K) \xrightarrow{d_{n+1}} C^{cw}_n(K)$ by:

$$\begin{tikzcd}
C^{cw}_{n+1}(K) \arrow[rr, "d_{n+1}"] \arrow[d, "\Psi_{n+1}", "\cong"'] & & C^{cw}_n(K) \\
H_{n+1}(K^{(n+1)}, K^{(n)}) \arrow[rr, "\beta_{n+1}"] \arrow[dr, "\partial_{n+1}"'] & & H_n(K^{(n)}, K^{(n-1)}) \arrow[u, "\Phi_n"', "\cong"] \\
& \tilde{H}_n(K^{(n)}) \arrow[ur, "j_n"'] &
\end{tikzcd}$$

Clearly, $d_n \circ d_{n+1} = 0$ because $\beta_n \circ \beta_{n+1} = 0$.

Let us write this differential explicitly. For $\sigma \in C^{cw}_{n+1}(K)$, write

$$d_{n+1}(\sigma) = \sum_{\tau \in I_n} \underbrace{[\tau : \sigma]}_{\in \mathbb{Z}} \cdot \tau$$

INCIDENCE NUMBER