THE FIRST HOMOLOGY GROUP

We now establish a link between the present subject of homology and our previous discussion of homotopy. In particular, what the connection between the fundamental group of a space & H_1 of a space is.

THEOREM [HUREWICZ THEOREM]

Let X be a path-connected space. Fix a base point $x_0 \in X$. Put

$$G := \pi_1(X, x_0)$$

Then

$$H_1(X) \cong G^{ab} := G/[G,G].$$

abelianization

Recall that $[G,G]$ is the subgroup generated by all the commutators, ie elements of the form $[g,h] = g^{-1}h^{-1}gh$.

Examples

(1) $\pi_1(S^n, *) \cong \begin{cases} 0 & n \geq 2 \\ \mathbb{Z} & n = 1 \end{cases}$

$\Rightarrow H_1(S^n) = \begin{cases} 0 & n \geq 2 \\ \mathbb{Z} & n = 1 \end{cases}$

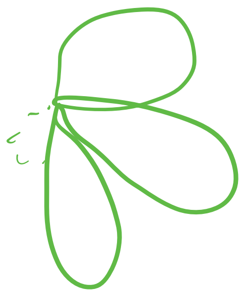
(2) Recall that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$

then $\pi_1(T) = \pi_1(S^1 \times S^1)$

$$\cong \pi_1(S^1) \times \pi_1(S^1)$$

$$\cong \mathbb{Z} \times \mathbb{Z}$$

$\Rightarrow H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$

3 $H_1\Big($  $\Big) \cong \big($ free non-abelian group on n letters $\big)^{ab}$

π_1

bouquet of n circles

$$\cong \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{n\text{-times}}$$

Proving the Hurewicz theorem

Some notation

- homotopy classes $\{\dots\}$, homology classes [...]

- $f \underset{\pi}{\simeq} g$ means that f & g are homotopic & $f \underset{H}{\simeq} g$ means homologous

Lemma 1

Let $f, g : I \to X$ be two paths with $f(1) = g(0)$. Consider the 1-chain

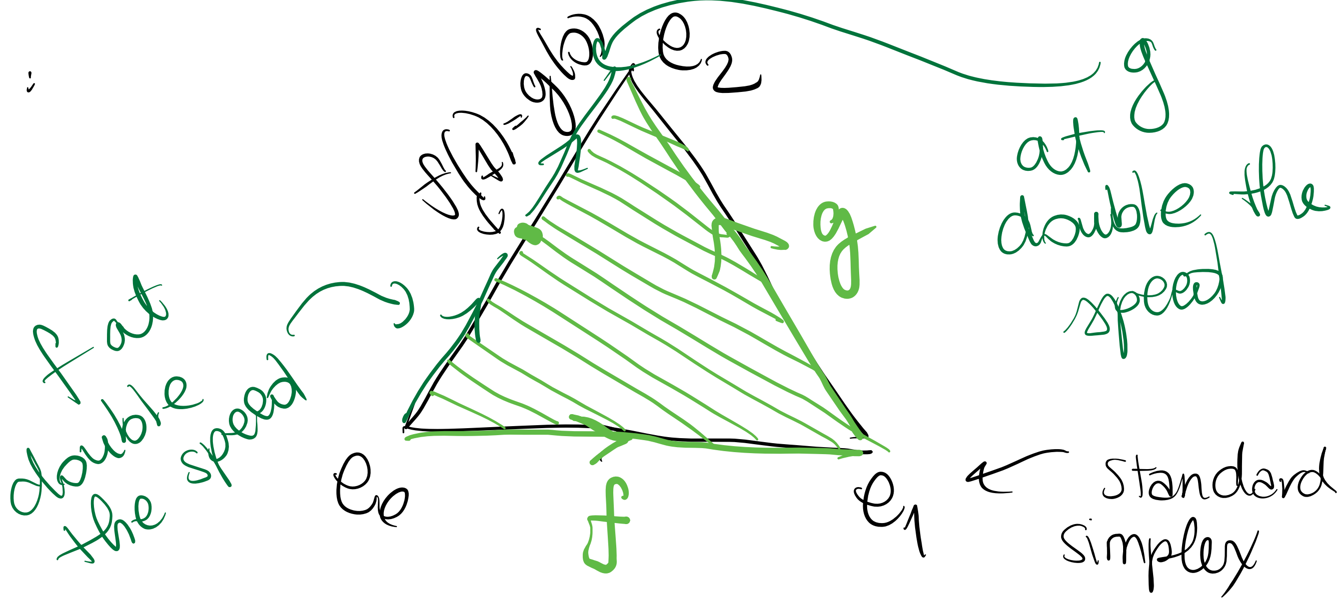
$$c := f * g - f - g \in S_1(X).$$

Then c is a boundary (hence $[c] = 0 \in H_1(X)$).

Proof of Lemma 1

Define $\sigma : \Delta^2 \to X$ as follows:

Δ^2:



- On the edge e_0, e_1 let it be f
- on the edge e_1, e_2 let it be g

Extend σ to the rest of Δ^2 s.t. on each segment in Δ^2, which is perpendicular to e_0e_2, σ is constant.

So on e_0e_2 we get that σ is $f * g$.

Let us calculate $\partial\sigma$:

$$\partial\sigma = (-1)^0 \sigma\big|_{[e_1,e_2]} + (-1)^1 \sigma\big|_{[e_0,e_2]} + (-1)^2 \sigma\big|_{[e_0,e_1]}$$

$$= g - f*g + f = f + g - f*g$$

$\Rightarrow$ $f*g - f - g$ is a boundary.

Lemma 2

① The constant path $c: I \to X$ is a boundary.

② Let $f: I \to X$ be a path. then the 1-chain $f + f^{-1}$ is a boundary.

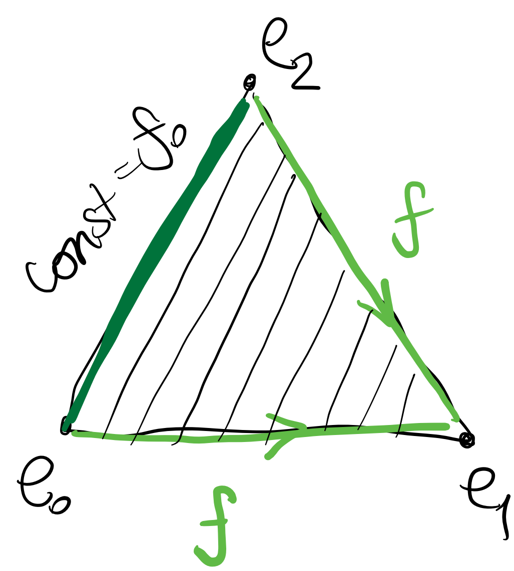
Proof of Lemma 2

① Define $\tau: \Delta^2 \to X$ to be the constant simplex (constant at the same point as c).

$\partial\tau = c - c + c = c$

(2) Define $\sigma: \Delta^2 \to X$ by defining σ to be on the edge e_0e_1 as well as e_2e_1.

Extend σ to the rest of Δ^2 by setting it to be constant on each segment parallel to e_0e_2.



$$\partial\sigma = f^{-1} - \text{const} + f$$

Since the constant edge is also a boundary by (1) $\partial\tau$,

$$\partial(\sigma+\tau) = f^{-1} + f$$

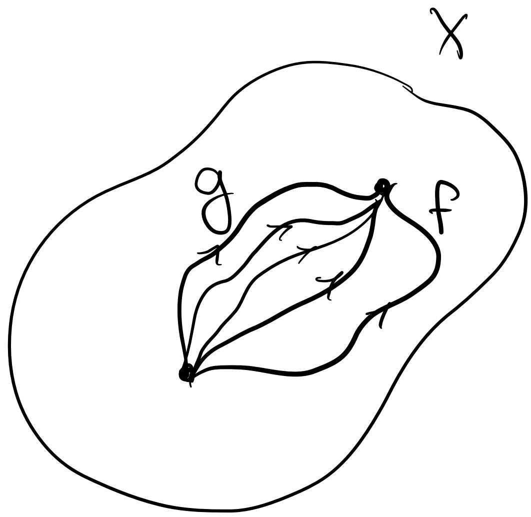
$f^{-1} + f$ is also a boundary.

Lemma 3

If $f, g : I \to X$ are paths with $f(0) = g(0)$, $f(1) = g(1)$ and $f \underset{\pi_1}{\simeq} g$ rel ∂I, then $f \underset{H}{\simeq} g$.

Proof

Let $F : I \times I \to X$ be a homotopy rel ∂I between f and g. We have

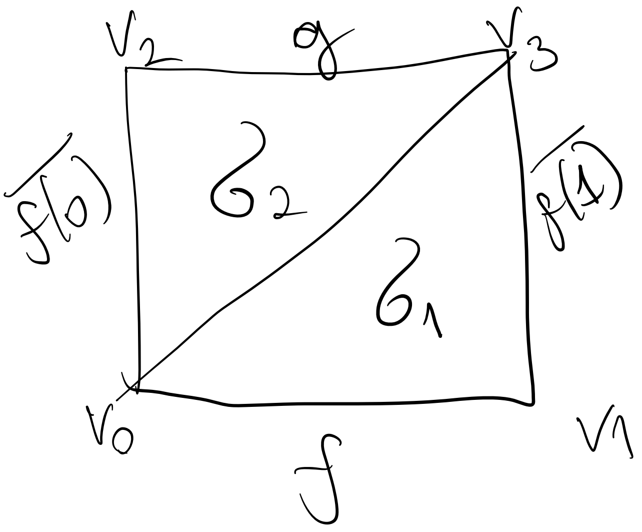


$$F\big|_{0 \times I} = \text{const} = f(0)$$

and

$$F\big|_{1 \times I} = \text{const} = f(1).$$

This homotopy yields a pair of



singular 2-simplices σ_1 & σ_2 in X.

$$\sigma_1\big|_{[v_0,v_3]} = \sigma_2\big|_{[v_0,v_3]}$$

$$\partial\sigma_1 = \sigma_1\big|_{[v_1,v_3]} - \sigma_1\big|_{[v_0,v_3]} + \sigma_1\big|_{[v_0,v_1]}$$

$$= \text{const}\,\overline{f(1)} - \sigma_1\big|_{[v_0,v_3]} + f$$

$$\partial\sigma_2 = \sigma_2\big|_{[v_2,v_3]} - \sigma_2\big|_{[v_0,v_3]} + \sigma_2\big|_{[v_0,v_2]}$$

$$= g - \sigma_2\big|_{[v_0,v_3]} + \text{const}\,\overline{f(0)}$$

We compute

$$\partial(\sigma_1 - \sigma_2) = \text{const}\,\overline{f(1)} - \cancel{\sigma_1\big|_{[v_0,v_3]}} + f$$

$$- g + \cancel{\sigma_2\big|_{[v_0,v_3]}} - \text{const}\,\overline{f(0)}$$

$$= f - g + \text{const}\,\overline{f(1)} - \text{const}\,\overline{f(0)}$$

Since constant singular simplices are

boundaries, so is $f-g$. This implies that $f \underset{H}{\simeq} g$. ∎

Now that we have proved these lemmas we return to the proof of the Hurewicz theorem.

First we need a map from $\pi_1(X, x_0)$ to $H_1(X)$:

$$\phi : \pi_1(X, x_0) \to H_1(X)$$

Let $\{f\} \in \pi_1(X, x_0)$ and let $f : I \to X$ be a loop representing $\{f\}$ in G.

f is a cycle since

$$\partial f = f(1) - f(0) = x_0 - x_0 = 0.$$

Define

$$\phi(\{f\}) := [f].$$

CLAIM: ϕ is well defined

This statement follows from Lemma 3.

Let $g \in \{f\}$. Then $f \underset{\Pi}{\simeq} g$ by definition.

By lemma 3 we also have that

$f \underset{H}{\simeq} g$, ie. $[f] = [g]$.

CLAIM: ϕ is a homomorphism of groups.

Let $f, g : I \to X$ be two loops based at x_0. Then

$$\phi(\{f\} * \{g\}) = \phi(\{f * g\})$$

$$= [f * g] \overset{*}{=} [f] + [g] = \phi(\{f\}) + \phi(\{g\})$$

*By Lemma 1 $[f * g] = [f] + [g]$.

Since $H_1(X)$ is abelian,
ϕ sends $[G,G]$ to 0.

$\Rightarrow \phi$ induces a homomorphism

$$\phi_* : G^{ab} \to H_1(X).$$

THEOREM [HUREWICZ]

ϕ_* is an isomorphism of groups.

Proof

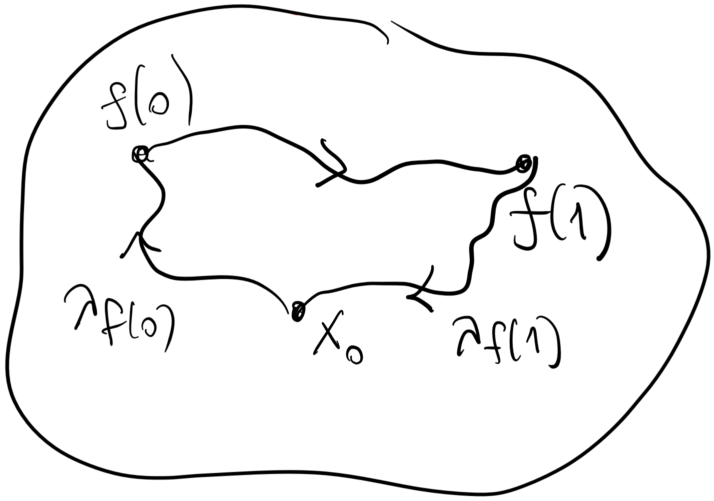
For all $x \in X$, choose in an arbitrary way a path λ_x from x_0 to x in such a way that $\lambda_{x_0} = \text{const}$.

Define a homomorphism

$$\Psi : S_1(X) \to G^{ab}$$

as follows:
for a generator of $S_1(X)$
$f: I \to X$, put

$$\Psi(f) = \{ \lambda_{f(0)} * f * \lambda_{f(1)}^{-1} \} \in G^{ab}$$



(I is identified with Δ^1)

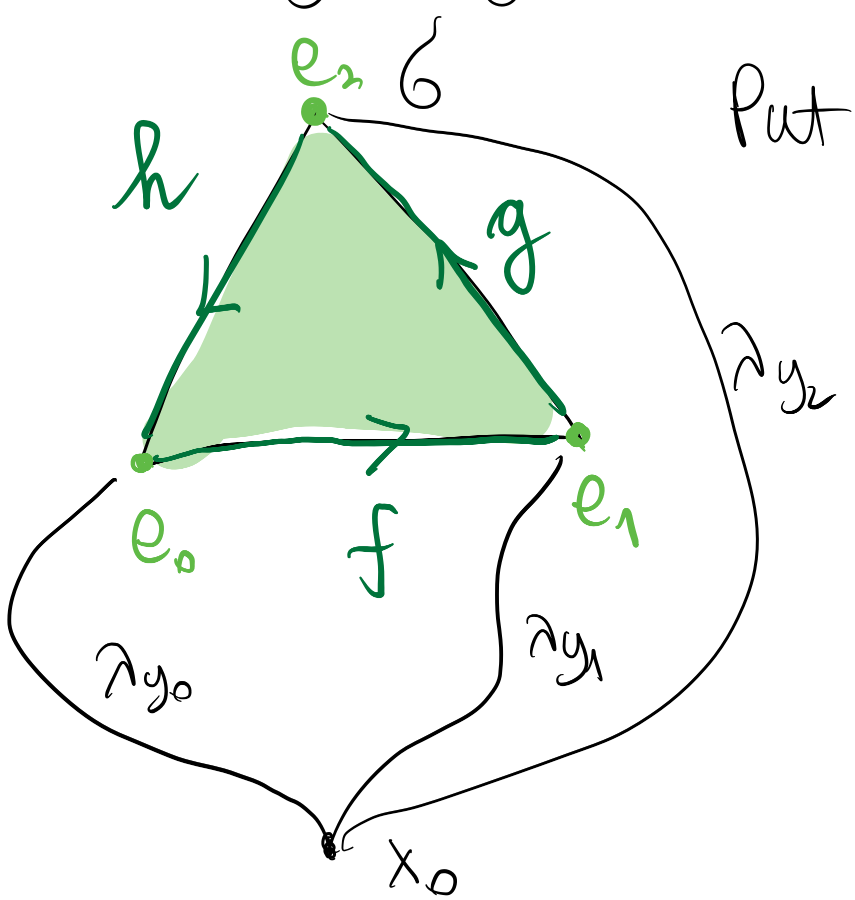
As $S_1(X)$ is free-abelian & G^{ab} is abelian, the above defines uniquely the homomorphism Ψ.

Lemma 4

$\forall b \in B_1(X)$ we have $\Psi(b) = 1 \in G^{ab}$.

Proof

Because Ψ is a homomorphism, it is enough to check that Ψ gets the value 1 on b's of the type $b = \partial\sigma$, where $\sigma: \Delta^2 \to X$ is any singular 2-simplex.



Put $y_i = \sigma(e_i)$

$f := \sigma|_{e_0 e_1}$

$g := \sigma|_{e_1 e_2}$

$h := \sigma|_{e_2 e_0}$

$$\Psi(\partial\sigma) = \Psi(g - h^{-1} + f) =$$

$$= \Psi(g) * (\Psi(h^{-1}))^{-1} * \Psi(f)$$

$$\overset{G^{ab}\text{ abelian}}{=} \Psi(f) * \Psi(g) * (\Psi(h^{-1}))^{-1}$$

$$= \{\lambda_{y_0} * f * \underbrace{\lambda_{y_1}^{-1} * \lambda_{y_1}}_{\substack{\text{homotopic}\\ \text{to const}\\ \text{rel } \partial I}} * g * \lambda_{y_2}^{-1} * (\lambda_{y_0} * h^{-1} * \lambda_{y_2}^{-1})^{-1}\}$$

$$= \{\lambda_{y_0} * f * g * \underbrace{\lambda_{y_2}^{-1} * \lambda_{y_2}}_{\substack{\simeq_{\partial I} \text{ const}\\ \text{rel } \partial I}} * h * \lambda_{y_0}^{-1}\}$$

$$= \{\lambda_{y_0} * f * g * h * \lambda_{y_0}^{-1}\} = (*)$$

Now $f * g * h \underset{\pi}{\simeq} \text{const}_{y_0}$ rel ∂I

$$(*) = \{\lambda_{y_0} * \lambda_{y_0}^{-1}\} = 1 \in G^{ab}$$

So far we have

ϕ, ϕ_*, Ψ.

Since $\Psi(B_1(X)) = \{1\}$, Ψ induces a homomorphism

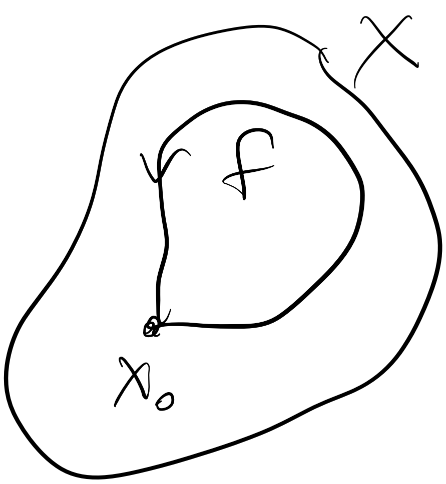
$$\Psi_* : H_1(X) \to G^{ab}.$$

(we restrict Ψ to Z_1)

CLAIM

$\Psi_* \circ \phi_* = id$

Proof



If f is a loop based in x_0, then

$$\Psi_* \circ \phi_*(f) = \Psi_*(\{f\}) =$$

$$= \{\underbrace{\lambda_{x_0}}_{\text{const}} * f * \underbrace{\lambda_{x_0}^{-1}}_{\text{const}}\} = \{f\}$$

CLAIM

$$\phi_* \circ \Psi_* = id$$

Proof

Note that $X \ni x \longmapsto \lambda_x$ induces a homomorphism

$$\begin{aligned} S_0(X) &\to S_1(X) \\ c &\mapsto \lambda_c \\ \sum_{x \in X} n_x x &\longmapsto \sum_{x \in X} n_x \lambda_x \end{aligned}$$

We will denote $\sum\limits_{x \in X} n_x \lambda_x$ by

$\lambda_{\sum\limits_{x \in X} n_x x}$.

Lemma 5

① Let $\sigma : I \to X$ be a 1-simplex (a path). Then

$$\phi_* \Psi(\sigma) = [\sigma + \lambda_{\sigma(0)} - \lambda_{\sigma(1)}] = $$
$$= [\sigma - \lambda_{\partial c}].$$

② If c is a 1-chain in X, then $\phi_* \Psi(c) = [c - \lambda_{\partial c}]$.

In particular, if c is a cycle, then $\phi_* \Psi(c) = [c]$.

Proof

(1) $\phi_* \Psi(\sigma) = \phi_* \{ \lambda_{\sigma(0)}^{*} * \sigma * \lambda_{\sigma(1)}^{-1} \} =$

use Lemmas 1 & 2

$= [\lambda_{\sigma(0)} * \sigma * \lambda_{\sigma(1)}^{-1}] =$

$= [\lambda_{\sigma(0)} + \sigma - \lambda_{\sigma(1)}] = [\sigma - \lambda_{\partial\sigma}]$

(2) Follows from the linearity of the map $S_0(X) \ni c \mapsto \lambda_c \in S_1(X)$ and other maps involved here.

If c is a cycle, $\partial c = 0$ &

$\phi_* \Psi(c) = [c - \lambda_{\partial c}] = [c - 0] = [c]$

COROLLARY

$\phi_* \Psi_* [c] = [c]$, ie $\phi_* \circ \Psi_* = id$.

This statement therefore completes the proof.

The next important property of singular homology is homotopy invariance.

HOMOTOPY INVARIANCE

Recall that a continuous map $f: X \to Y$ induces a chain map $f_c : S_n(X) \to S_n(Y)$ between chain complexes $C(X)$ and $C(Y)$ and f_c in turn induces a map $f_* : H_n(X) \to H_n(Y)$. We have already proved that if f is a homeomorphism, then f_* is an isomorphism. Now we turn our attention to maps between homology groups induced by homotopic maps. In particular, we can prove the following theorem [For this part we follow Hatcher]

THEOREM

If two maps $f, g : X \to Y$ are homotopic, then they induce the same homomorphism

$$f_* = g_* : H_n(X) \to H_n(Y)$$

In particular, if f is a homotopy equivalence, then f_* is an isomorphism for all n.

Proof

The essential ingredient of the proof is to subdivide $\Delta^n \times I$ into simplices.

For a general n:

Let $\Delta^n \times \{0\} = [v_0, \dots, v_n]$

and $\Delta^1 \times \{1\} = [w_0, \dots, w_1]$,

where v_i and w_i have the same image under the projection $\Delta^n \times I \to \Delta^n$.

We pass from $[v_0, \dots, v_n]$ to $[w_0, \dots, w_n]$ by interpolating a sequence of n-simplices each obtained from the preceding one by moving one vertex v_i up to w_i, starting with v_n and working backwards to v_0.

First step: $[v_0, \ldots, v_n] \to [v_0, \ldots, v_{n-1}, w_n]$

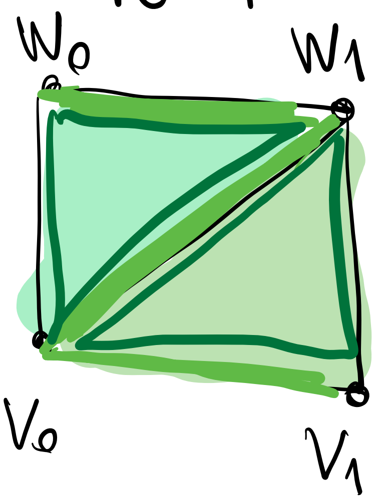
Second step: $[v_0, \ldots, v_{n-1}, w_n] \to [v_0, \ldots, w_{n-1}, w_n]$

$\vdots$

$[v_0, \ldots, v_i, w_{i+1}, \ldots, w_n] \to [v_0, \ldots, v_{i-1}, w_i, \ldots, w_n]$

The region between these two simplices is exactly the $(n+1)$-sx $[v_0, \ldots, v_i, w_i, \ldots, w_n]$ which has $[v_0, \ldots, v_i, w_{i+1}, \ldots, w_n]$ as a lower face and $[v_0, \ldots, v_{i-1}, w_i, \ldots, w_n]$ as an upper face.

$n = 1$

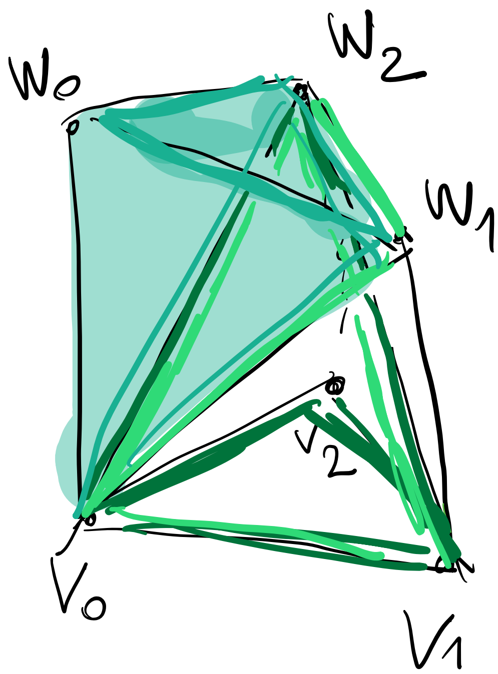


$$\left.\begin{array}{l} [v_0, v_1] \to \\ [v_0, w_1] \to \\ [w_0, w_1] \end{array}\right\} \text{sequence of 1-simplices}$$

Regions in between that are

2-simplices: $[v_0, v_1, w_1]$, $[v_0, w_0, w_1]$

$n=2$



$[v_0, v_1, v_2] \to$

$[v_0, v_1, w_2] \to$

$[v_0, w_1, w_2] \to$

$[w_0, w_1, w_2]$

Regions in between that are

3-simplices: $[v_0, v_1, v_2, w_2]$

$[v_0, v_1, w_1, w_2]$

$[v_0, w_0, w_1, w_2]$

Altogether, $\Delta^n \times I$ is the union of the $(n+1)$-simplices $[v_0, \dots, v_i, w_i, \dots, w_n]$, each intersecting the next in an n-simplex face.

Given a homotopy $F: X \times I \to Y$ from f to g we define

$$P_n: S_n(X) \to S_{n+1}(Y),$$

a homomorphism of groups given on generators by the following formula:

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i \underbrace{F \circ (\sigma \times id_I)\Big|_{[v_0, \dots, v_j, w_j, \dots, w_n]}}_{\text{these are singular (n+1)-simplices}}$$

this operator is called the

PRISM OPERATOR.

We will show that prism operators satisfy the basic relation

$$\partial P = g_c - f_c - P\partial$$

top, bottom & the sides of the prism

geometrically ∂P represents the boundary of the prism

To prove this relation we calculate

$$\partial P(\sigma) = \sum_{j\le i} (-1)^i(-1)^j F\circ(\sigma\times id_I)\big|_{[v_0,\dots,\hat{v}_j,\dots,v_i,w_i,\dots,w_n]}$$

$$+ \sum_{j\ge i} (-1)^i(-1)^{j+1} F\circ(\sigma\times id_I)\big|_{[v_0,\dots,v_i,w_i,\dots,\hat{w}_j,\dots,w_n]}$$

The terms with $i=j$ in the two sums cancel except for

$F \circ (\sigma \times id_I)|_{[\hat{v}_0, w_0, \dots, w_n]}$, which

is $g \circ \sigma = g_c(\sigma)$, and $-F \circ (\sigma \times id_I)|_{[v_0, \dots, v_n, \hat{w}_n]}$

which is $-f \circ \sigma = -f_c(\sigma)$.

The terms with $i \neq j$ are exactly $-P\partial(\sigma)$ since

$$P\partial(\sigma) =$$

$$\sum_{i<j} (-1)^i (-1)^j F \circ (\sigma \times id_I)|_{[v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n]} +$$

$$\sum_{i>j} (-1)^{i-1} (-1)^j F \circ (\sigma \times id_I)|_{[v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots, w_n]}$$

Now we finish the proof of the theorem.

If $c \in S_n(X)$ is a cycle, then

$$g_c(c) - f_c(c) = \partial P(c) + P\partial(c)$$
$$= \partial P(c)$$

Since $\partial c = 0$. This means that $g_c(c) - f_c(c)$ is a boundary and so $[g_c(c)] = [f_c(c)]$. This implies that g_* equals f_* on the homology class of c.

$\blacksquare$