Quick Intro to Homological Algebra

Let us first recall the definition of a chain complex & homology of a general chain complex.

Definition

A **CHAIN COMPLEX** is a sequence of abelian groups C_i, $i \in \mathbb{Z}$, together with a sequence of homomorphisms $\partial_i : C_i \to C_{i-1}$ s.t. $\partial_{i-1} \circ \partial_i = 0 \quad \forall i$ (sometimes written as $\partial \circ \partial = 0$).

∂ is called the **BOUNDARY OPERATOR**.

$$\cdots \longrightarrow C_{i+1} \xrightarrow{\partial} C_i \xrightarrow{\partial} C_{i-1} \xrightarrow{\partial} C_{i-2} \xrightarrow{\partial} \cdots$$

Let $Z_i = \ker(C_i \xrightarrow{\partial} C_{i-1})$ ← cycles and

$$B_i = \mathrm{Im}\left(C_{i+1} \xrightarrow{\partial} C_i\right)$$ ← boundaries

Since $\partial \circ \partial = 0$, we have $B_i \subset Z_i$.

Define $H_i(\mathcal{C}_\bullet) := Z_i / B_i$.

↑ homology in degree i

$\mathcal{C}_\bullet = (C_\bullet, \partial_\bullet)$

Definition [morphism of chain complexes]

If $\mathcal{A}_\bullet = (A_\bullet, \partial^A_\bullet)$, $\mathcal{B}_\bullet = (B_\bullet, \partial^B_\bullet)$ are chain complexes, a **CHAIN MAP**

$f : \mathcal{A}_\bullet \to \mathcal{B}_\bullet$ is a

collection of homomorphisms

$f : A_i \to B_i \ \forall i$ s.t. $f \circ \partial^A = \partial^B \circ f$

$$\begin{array}{ccccccccc}
\cdots & A_{i+1} & \xrightarrow{\partial_A} & A_i & \xrightarrow{\partial_A} & A_{i-1} & \to & \cdots \\
 & \downarrow f & © & \downarrow f & © & \downarrow f & & \\
\cdots & B_{i+1} & \xrightarrow{\partial_B} & B_i & \xrightarrow{\partial_B} & B_{i-1} & \to & \cdots
\end{array}$$

all squares are commutative

PROPOSITION

Let $f: \mathcal{A}_\bullet \to \mathcal{B}_\bullet$ be a chain map. Then f induces a homomorphism

$$f_*: H_i(\mathcal{A}_\bullet) \to H_i(\mathcal{B}_\bullet)$$

for all $i \in \mathbb{Z}$ by the following procedure:

Let $\alpha \in H_i(\mathcal{A}_\bullet)$. Pick a cycle $a \in A_i$ s.t. $[a] = \alpha$. Define

$$f_*(\alpha) = [f(a)].$$

Moreover, if $\mathcal{A}_\bullet, \mathcal{B}_\bullet$ and $\mathcal{C}_\bullet$ are chain complexes and $f: \mathcal{A}_\bullet \to \mathcal{B}_\bullet$, $g: \mathcal{B}_\bullet \to \mathcal{C}_\bullet$ are chain maps, then $g \circ f$ is also a chain map and $(g \circ f)_* = g_* \circ f_*$ and $(id_{\mathcal{A}_\bullet})_* = id_{H_i(\mathcal{A}_\bullet)}$ for all i.

Key ingredient: Chain maps map boundaries to boundaries and cycles to cycles.

EXACT SEQUENCES

Let A, B, C be abelian groups, and

$A \xrightarrow{i} B \xrightarrow{j} C$ be two homomorphisms.

The sequence $A \xrightarrow{i} B \xrightarrow{j} C$ is called **EXACT** if $\ker j = \operatorname{Im} i$.

A sequence $\cdots \to A_{k+1} \xrightarrow{f_{k+1}} A_k \xrightarrow{f_k} A_{k-1} \xrightarrow{f_{k-1}} \cdots$

is called **EXACT** if $A_{k+1} \xrightarrow{f_{k+1}} A_k \xrightarrow{f_k} A_{k-1}$

is EXACT for all k.

Remark

① $0 \to A \xrightarrow{f} B$ is exact $\Leftrightarrow$

f is injective ($\ker f = \{0\}$)

② $A \xrightarrow{g} B \to 0$ is exact $\Leftrightarrow$

g is surjective ($\operatorname{Im} g = \operatorname{Ker} 0 = B$)

③ $0 \to A \xrightarrow{h} B \to 0$ is exact $\Leftrightarrow$

h is an isomorphism.

④ If $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is exact, the embedding $i : A \hookrightarrow B$ and the surjection j induce an isomorphism $B/i(A) \xrightarrow{\cong} C$

(this holds since j induces an isomorphism $B/\operatorname{Ker} j \to \operatorname{Im} j$)

$$\begin{array}{ccc} B/\operatorname{Ker} j & \longrightarrow & \operatorname{Im} j \\ \parallel & & \parallel \\ B/i(A) & & C \end{array}$$

An exact sequence $0 \to A \to B \to C \to 0$ is called a **SHORT EXACT SEQUENCE (SES)**.

Let $\mathcal{A}_\bullet, \mathcal{B}_\bullet, \mathcal{C}_\bullet$ be chain complexes. Let $i: \mathcal{A}_\bullet \to \mathcal{B}_\bullet$, $j: \mathcal{B}_\bullet \to \mathcal{C}_\bullet$ be chain maps. We can look at the sequence

$$0 \to \mathcal{A}_\bullet \xrightarrow{i} \mathcal{B}_\bullet \xrightarrow{j} \mathcal{C}_\bullet \to 0 \quad (*)$$

We say that this sequence is exact if $\forall n \in \mathbb{Z}$ $0 \to A_n \xrightarrow{i} B_n \xrightarrow{j} C_n \to 0$

is exact.

We call $(*)$ a **SES of chain complexes**.

THEOREM

Let $0 \to \mathcal{A}_\bullet \xrightarrow{i} \mathcal{B}_\bullet \xrightarrow{j} \mathcal{C}_\bullet \to 0$

be a SES of chain complexes. Then

it induces a **LONG EXACT SEQUENCE IN HOMOLOGY**

$$\cdots \to H_{n+1}(\mathcal{A}_\bullet) \xrightarrow{i_*} H_{n+1}(\mathcal{B}_\bullet) \xrightarrow{j_*} H_{n+1}(\mathcal{C}_\bullet) \xrightarrow{\partial_*}$$

$$\to H_n(\mathcal{A}_\bullet) \xrightarrow{i_*} H_n(\mathcal{B}_\bullet) \xrightarrow{j_*} H_n(\mathcal{C}_\bullet) \xrightarrow{\partial_*}$$

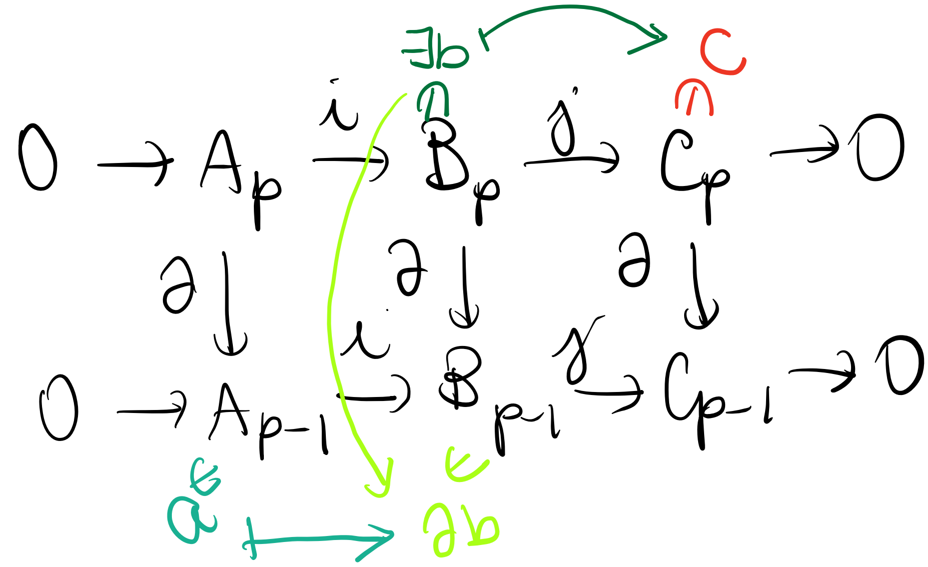
$$\to H_{n-1}(\mathcal{A}_\bullet) \xrightarrow{i_*} H_{n-1}(\mathcal{B}_\bullet) \xrightarrow{j_*} H_{n-1}(\mathcal{C}_\bullet) \xrightarrow{\partial_*} \cdots$$

The homomorphism $\partial_* : H_n(\mathcal{C}_\bullet) \to H_{n-1}(\mathcal{A}_\bullet)$

is called the **CONNECTING HOMOMORPHISM**.

Proof

Let's examine what happens on the chain level in degrees p and $p-1$:



We will define $\partial_* : H_p(\mathcal{C}_\bullet) \to H_{p-1}(\mathcal{A}_\bullet)$ as follows.

Let $\mathfrak{m} \in H_p(\mathcal{C}_\bullet)$. Choose a cycle $c \in C_p$

(ie. $\partial c = 0$) s.t. $[c] = \mathfrak{m}$.

$B_p \xrightarrow{j} C_p$ is a surjection, so $\exists b \in B_p$ s.t. $j(b) = c$.

Since $j\partial(b) = \partial(j(b)) = \partial c = 0$,

$\partial(b) \in \ker j = \operatorname{Im} i$.

$\Rightarrow \exists!\ a \in A_{p-1}$ s.t. $i(a) = \partial b$.

Note that

$$i(\partial a) = \partial i(a) = \partial(\partial b) = 0.$$

But i is injective, hence $\partial a = 0$,

ie. a is a cycle.

Define $\partial_*(m) := [a]$.

CLAIM: the definition of ∂_* is good, ie. it doesn't depend on the choice of c (with $[c] = m$) nor on the choice of b.

Proof of claim:

Fix first c and suppose that $c = j(b')$. Define a' as before but using b'.

$j(b-b') = 0$ (since $j(b) = j(b') = c$)

$\Rightarrow b - b' \in \ker j = \operatorname{Im} i$, so

$b - b' = i(a_0)$ for some $a_0 \in A_p$.

$\partial b - \partial b' = \partial i(a_0) = i(\partial a_0)$

But $\partial b - \partial b' = i(a) - i(a') = i(a-a')$.

So $i(a-a') = i(\partial a_0)$. Since i is injective, $a - a' = \partial a_0 \Rightarrow [a] = [a']$.

We'll show next that the definition of $\partial_* m$ is independent of c (with $[c] = m$).

Consider another cycle $c' \in C_p$ with $[c'] = m$.

$\Rightarrow c' = c + \partial c''$.

Since j is surjective, we may choose b with $j(b) = c$, and b'' with $j(b'') = c''$.

Put

$$b' := b + \partial b''$$

$$j(b') = j(b) + j(\partial b'') = c + \partial j(b'') =$$

$$= c + \partial c'' = c'$$

So b' serves as an element that is sent to c' by j. Now consider $\partial b'$, then take the unique $a' \in A_{p-1}$ with $i(a') = \partial b'$. But

$$\partial b' = \partial b + \partial \partial b'' = \partial b.$$

So

$$i(a') = \partial b' = \partial b = i(a).$$

Since i is injective, $a' = a$.

This completes the proof of the claim.

CLAIM: ∂_* is a homomorphism.

Proof of claim:

Let $c^1, c^2 \in C_p$ be two cycles.

From the recipe for ∂_*, we choose $b^1, b^2 \in B_p$, $a^1, a^2 \in A_{p-1}$ with $i(a^1) = \partial b^1$,

$i(a^2) = \partial b^2$.

Then $\partial_*[c^1] = [a^1]$, $\partial_*[c^2] = [a^2]$.

To apply ∂_* on $[c^1] + [c^2] = [c^1 + c^2]$ we can choose $c^1 + c^2$ to be the representative of $[c^1] + [c^2]$.

$j(b^1+b^2)=j(b^1)+j(b^2)=c^1+c^2$ and

$i(a^1+a^2)=i(a^1)+i(a^2)=\partial b^1+\partial b^2=$

$=\partial(b^1+b^2),$

$\Rightarrow \partial_*([c^1]+[c^2])=[a^1+a^2]=[a^1]+[a^2]$

$=\partial_*[c^1]+\partial_*[c^2].$

This proves the claim.

PROOF that the long sequence

$$\cdots \to H_{n+1}(\mathcal{A}_\cdot) \xrightarrow{i_*} H_{n+1}(\mathcal{B}_\cdot) \xrightarrow{j_*} H_{n+1}(\mathcal{C}_\cdot) \xrightarrow{\partial_*}$$

$$\to H_n(\mathcal{A}_\cdot) \xrightarrow{i_*} H_n(\mathcal{B}_\cdot) \xrightarrow{j_*} H_n(\mathcal{C}_\cdot) \xrightarrow{\partial_*}$$

$$\to H_{n-1}(\mathcal{A}_\cdot) \xrightarrow{i_*} H_{n-1}(\mathcal{B}_\cdot) \xrightarrow{j_*} H_{n-1}(\mathcal{C}_\cdot) \xrightarrow{\partial_*} \cdots$$

is exact.

We must verify six statements:

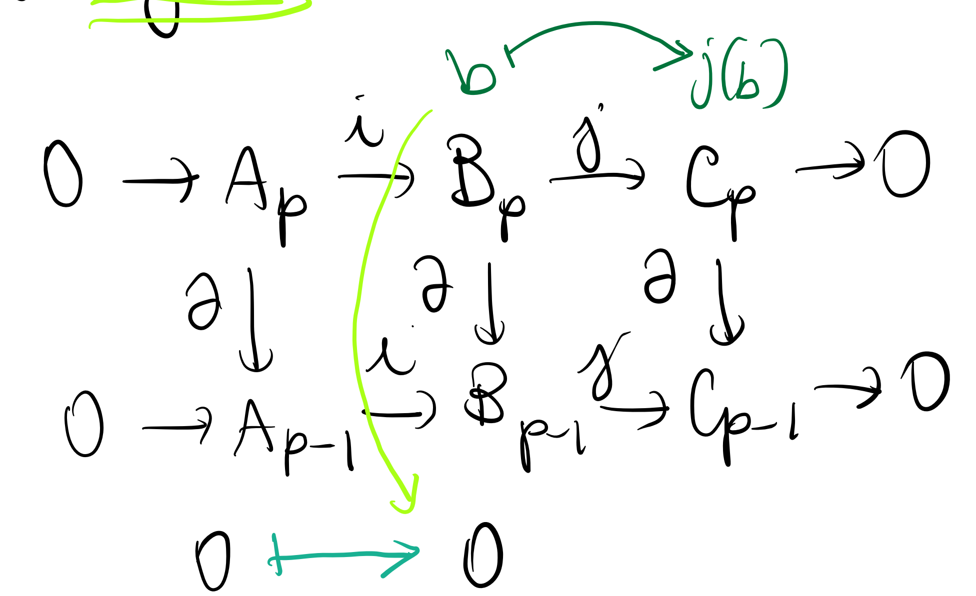
$\operatorname{Im} i_* \subseteq \ker j_*$, $\ker j_* \subseteq \operatorname{Im} i_*$,
$\operatorname{Im} j_* \subseteq \ker \partial_*$, $\ker \partial_* \subseteq \operatorname{Im} j_*$,
$\operatorname{Im} \partial_* \subseteq \ker i_*$ and $\ker i_* \subseteq \operatorname{Im} \partial_*$.

① $\operatorname{Im} i_* \subset \ker j_*$

This follows since $ji = 0$ implies that $j_* i_* = 0$.

② $\operatorname{Im} j_* \subset \ker \partial_*$

Let $\beta \in H_p(\mathcal{B}_\cdot)$ and let $b \in B_p$ be a cycle with $\beta = [b]$.



By construction $\partial_*(j_*[b]) = 0$

③ $\operatorname{Im} \partial_* \subseteq \ker i_*$

We must show that $i_* \partial_* = 0$.

Assume that $c \in C_p$ is a cycle.

$i_* \partial_* [c] = [i(a)]$, where $a \in A_{p-1}$ is such that $i(a) = \partial b$, where $j(b) = c$. $\Rightarrow$

$$i_* \partial_* [c] = [i(a)] = [\partial b] = 0.$$

④ $\ker j_* \subset \operatorname{Im} i_*$:

Assume that $j_*[b] = 0$, where b is a cycle. Since $j_*[b] = [j(b)]$, it follows that $j(b) = \partial c$ for some $c \in C_{p+1}$. Pick b' with $j(b') = c$.

Note that

$$j(b-\partial b') = j(b) - j(\partial b') =$$

$$= \partial c - \partial(j(b')) = \partial c - \partial c = 0$$

By exactness of the SES

$\exists a$ s.t.

$$i(a) = b - \partial b'.$$

Let us show that a is a cycle:

$$\partial i(a) = \partial b - \partial\partial b' = 0 - 0 = 0$$

$$\parallel$$

$$i(\partial a)$$

Since i is injective, $\partial a = 0$.

$$\Rightarrow i_*([a]) = [i(a)] = [b - \partial b']$$

$$= [b]$$

$$\Rightarrow [b] \in \operatorname{Im}(i_*).$$

⑤ $\ker i_* \subset \operatorname{Im} \partial_*$

Suppose that $i_*[a]=0$. $\Rightarrow$ $i(a)=\partial b$ for some $b\in B_p$.

Put $c:=j(b)$. We have

$$\partial(c)=\partial j(b)=j\partial(b)=ji(a)=0.$$

$\Rightarrow$ c is a cycle. Now by the definition of ∂_*, $\partial_*[c]=[a]$.

⑥ $\ker \partial_* \subset \operatorname{Im} j_*$

Suppose $\partial_*[c]=0$ for some cycle $c\in C_p$. Choose $b\in B_p$ with $j(b)=c$, and $a\in A_{p-1}$ with $i(a)=\partial b$.

Since $\partial_*[c]=[a]=0$, $a=\partial a'$

for some $a' \in A_p$.

Now: $\partial i(a') = i\partial a' = i(a) = \partial b$

$\Rightarrow \partial(b - i(a')) = 0$

It follows that $b - i(a')$ is a cycle.

We have:

$$j(b - i(a')) = j(b) - \underbrace{j \circ i}_{0}(a') = j(b) = c$$

and from here

$$j_*[b - i(a')] = [c].$$

∎

This kind of method of proof is called DIAGRAM CHASING.

ADDENDUM to theorem SES ⇒ LES

Let

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{A}_\bullet & \xrightarrow{i} & \mathcal{B}_\bullet & \xrightarrow{j} & \mathcal{C}_\bullet & \to & 0 \\
 & & \downarrow f & & \downarrow g & & \downarrow h & & \\
0 & \to & \mathcal{A}'_\bullet & \xrightarrow{i'} & \mathcal{B}'_\bullet & \xrightarrow{j'} & \mathcal{C}'_\bullet & \to & 0
\end{array}$$

be two SES of chain complexes and f, g, h chain maps s.t. the diagram above commutes. For $\forall p$ we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & A_p & \xrightarrow{i} & B_p & \xrightarrow{j} & \mathcal{C}_p & \to & 0 \\
 & & \downarrow f & \text{(I)} & \downarrow g & \text{(II)} & \downarrow h & & \\
0 & \to & A'_p & \xrightarrow[i']{} & B'_p & \xrightarrow[j']{} & \mathcal{C}'_p & \to & 0
\end{array}$$

Then we obtain two LES in homology with maps between them that makes all the squares commutative

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{\partial_*} & H_p(\mathcal{A}_\bullet) & \xrightarrow{i_*} & H_p(\mathcal{B}_\bullet) & \xrightarrow{j_*} & H_p(\mathcal{C}_\bullet) & \xrightarrow{\partial_*} & H_{p-1}(\mathcal{A}_\bullet) & \to \cdots \\
 & \downarrow f_* & (1) & \downarrow g_* & (2) & \downarrow h_* & (3) & \downarrow f_* & \\
\cdots \xrightarrow{\partial_*} & H_p(\mathcal{A}'_\bullet) & \xrightarrow{i'_*} & H_p(\mathcal{B}'_\bullet) & \xrightarrow{j'_*} & H_p(\mathcal{C}'_\bullet) & \xrightarrow{\partial_*} & H_{p-1}(\mathcal{A}'_\bullet) & \to \cdots
\end{array}$$

Proof

(1) & (2) commute because (I) & (II) on the chain level commute.

To prove commutativity of (3):

Let $c \in \mathcal{C}_p$ be a cycle.

$\partial_*[c] = [a]$, where $a \in A_{p-1}$ with

$i(a) = \partial(b)$, where $j(b) = c$.

Consider $f(a)$. Firstly,

$$i' \circ f(a) = g \circ i(a) = g \circ \partial(b) = \partial\, g(b)$$

and

$$j' \circ g(b) = h \circ j(b) = h(c).$$

} definition of ∂_*

It follows that

$$\partial'_* \circ h_*[c] = \partial'_*[h(c)] = [f(a)] = f_*[a] = f_* \partial_* [c].$$

∎

THE 5-LEMMA

Let

$$\begin{array}{ccccccccc} A & \to & B & \to & C & \to & D & \to & E \\ a\downarrow & & b\downarrow & & \downarrow c & & \downarrow d & & \downarrow e \\ A' & \to & B' & \to & C' & \to & D' & \to & E' \end{array}$$

be a commutative diagram of abelian groups with exact rows.

① If b&d are injective & a is surjective $\Rightarrow$ c is injective.

② If b&d are surjective & e is injective $\Rightarrow$ c is surjective.

③ If a, b, d, e are isomorphisms $\Rightarrow$ c is an isomorphism

Definition

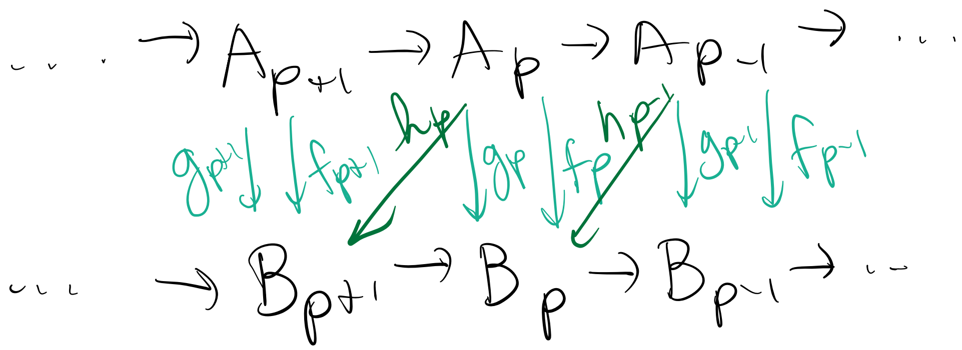
Let $f, g : A_\bullet \to B_\bullet$ be chain maps.

A **CHAIN HOMOTOPY** from f to g is a sequence of homomorphisms h_*

$$h_p : A_p \to B_{p+1} \;, \; p \in \mathbb{Z}$$

for which

$$\partial_{p+1} \circ h_p + h_{p-1} \circ \partial_p = g_p - f_p$$



Example

Let $f, g : X \to Y$ be continuous maps

$P: C_k(X) \to C_{k+1}(Y)$ (prism operator)

P is a chain homotopy from f_c to g_c.

A chain map $f: \mathcal{A}_\bullet \to \mathcal{B}_\bullet$ is a **CHAIN EQUIVALENCE** if there exists a chain map $g: \mathcal{B}_\bullet \to \mathcal{A}_\bullet$ and chain homotopies from $f \circ g$ to id and from $g \circ f$ to id.

Two chain complexes are chain equivalent if there exists a chain equivalence between them.

Proposition

If $f, g : A_\bullet \to B_\bullet$ are chain homotopic, then $f_* = g_*$.

Proof

Let $f \overset{h}{\simeq} g$.

Let a be a cycle, ie $\partial a = 0$.

Then

$$g(a) - f(a) = \partial(h(a)) + h(\underbrace{\partial a}_{=0})$$

$$= \partial(h(a))$$

↑ this is a boundary

$$\Rightarrow [g(a)] = [f(a)] \Rightarrow g_* = f_*.$$