RETRACTIONS, DEFORMATION RETRACTIONS

Definition

Let X be a space and $A \subset X$. A **RETRACTION** $r : X \to A$ is a map s.t. $r(a) = a$ $\forall a \in A$.
We say that A is a **RETRACT** of X.
A subspace A of X is called a **STRONG DEFORMATION RETRACT** of X if there exists a homotopy $F : X \times I \to X$ (called a **DEFORMATION**) such that

DEFORMATION RETRACTION

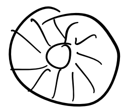
$$F(x,0) = x$$
$$F(x,1) \in A$$
$$F(a,t) = a \text{ for } a \in A \text{ and all } t \in I.$$

It is called a **DEFORMATION RETRACT** if the last equation is required only for $t = 1$.

Comment: A deformation retract A of a space X is homotopically equivalent to X.

Example

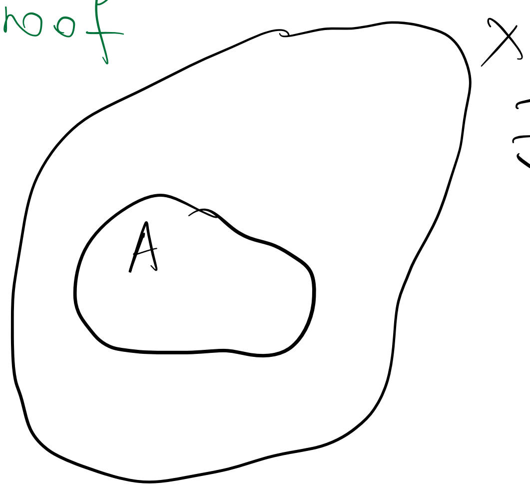
(1) $\{0\} \subset \mathbb{R}^n$ is a strong deformation retract.

(2) S^1 is a strong deformation retract of A ()

Proposition

If $A \subset X$ is a deformation retract, then $X \simeq A$.

Proof



$A \subset X$ def. ret.

$\exists \; F: X \times I \to X$

$F(x,0) = id$

$F(x,1) \in A$ for $\forall x \in X$

$F(a,1) = a$ for $a \in A$.

$i: A \hookrightarrow X$

$F(-,1): X \to A$

$F(-,1) \circ i = id_A$ by def. of F & i

$i \circ F(-,1) = F(-,1) \simeq id$
by def.

So $X \simeq A$.

PAIRS OF SPACES

Definition

Let X, Y be topological spaces and

$A \subset X$ & $B \subset Y$.

$f : (X, A) \to (Y, B)$ means

$f : X \to Y$ such that $f(A) \subset B$.

Let $f_0, f_1 : (X, A) \to (Y, B)$ be maps of pairs. We say they are homotopic if $\exists$ $F : X \times I \to Y$ with $F(x,0) = f_0(x)$, $F(x,1) = f_1(x)$ $\forall x \in X$ and such that

$F(a,t) \in B \quad \forall a \in A, t \in I.$

Definition

$A \subset X$ subspace. A **HOMOTOPY** $F: X \times I \to Y$ is called **RELATIVE to A** if $F(a,t)$ is independent of t $\forall a \in A$.

If $f_0 = F(-,0)$, $f_1 = F(-,1)$ we write

$$f_0 \underset{\text{rel.} A}{\simeq} f_1.$$

Example

$i: A \to X$
$r: X \to A$

A strong deformation retraction i.e. $r \simeq id_X$ is a homotopy relative to the subspace A.

OPERATIONS WITH HOMOTOPIES

Definition

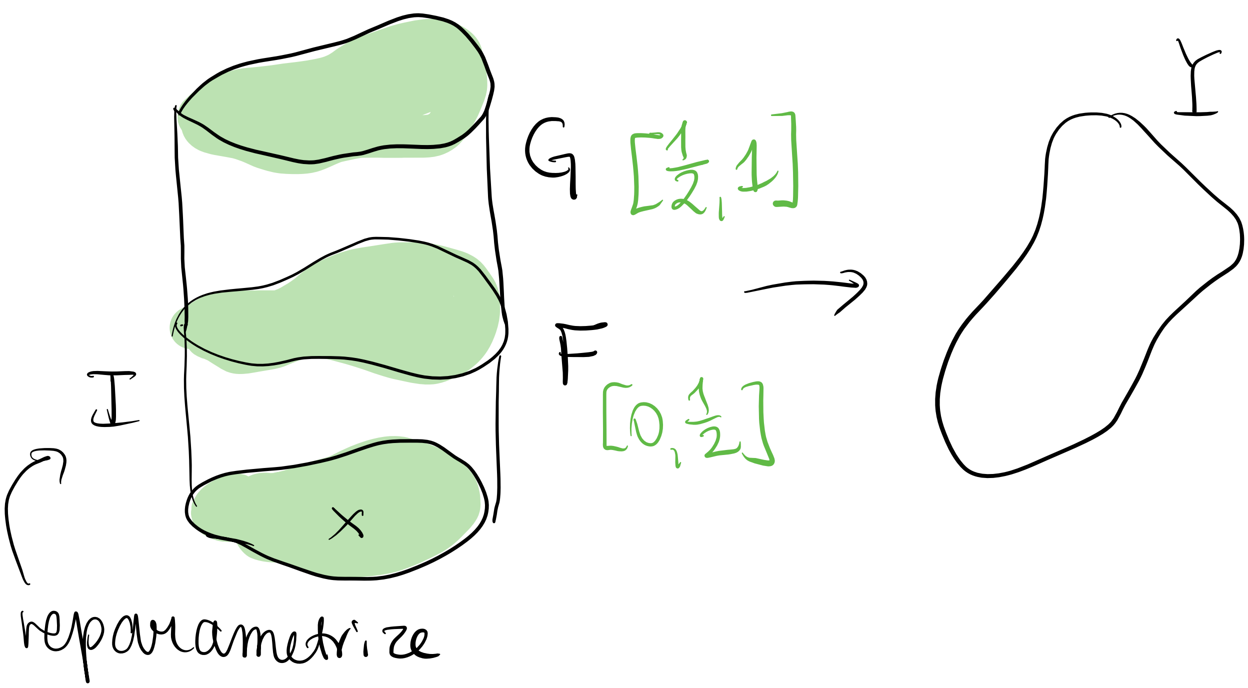
Let $F: X \times I \to Y$, $G: X \times I \to Y$ be two homotopies, $G(x,0) = F(x,1)$ $\forall x \in X$.

Define a new homotopy, **CONCATENATION**,

$$F * G : X \times I \longrightarrow Y$$

(concatenation of F & G)

$$F * G(x,t) = \begin{cases} F(x,2t) & 0 \le t \le \frac{1}{2} \\ G(x,2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$



One does not need to combine these homotopies at $t = \frac{1}{2}$. We can do it at any point and with arbitrary speed.

Definition

Let $\phi_1, \phi_2 : (I, \partial I) \to (I, \partial I)$

s.t. $\phi_1|_{\partial I} = \phi_2|_{\partial I}$ $\left(\begin{array}{l} \phi_1(0) = \phi_2(0) \\ \phi_1(1) = \phi_2(1) \end{array} \right)$

Let $F : X \times I \to Y$ be a homotopy.

Define $G_1(x,t) = F(x, \phi_1(t))$

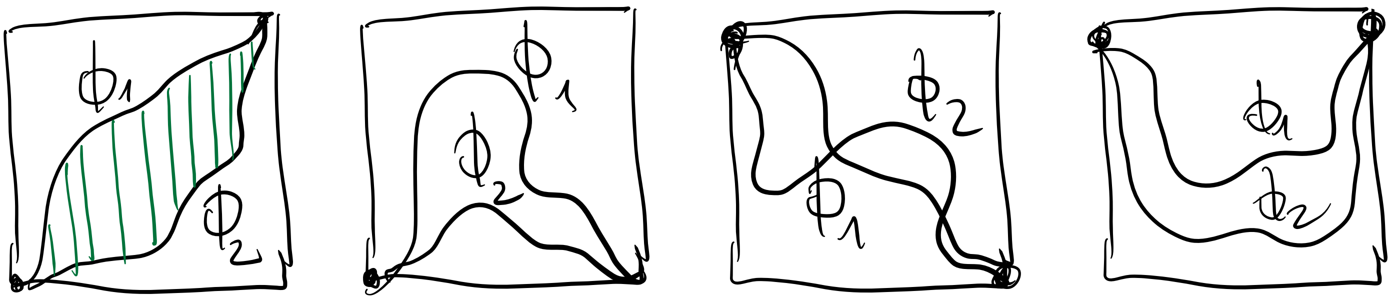
$G_2(x,t) = F(x, \phi_2(t))$.

REPARAMETRIZATIONS OF F

Proposition

$G_1 \simeq G_2$ rel $(X \times \partial I)$.

Proof



In each of these 4 cases we can use the straight line homotopy:

$$s\,\phi_2(t) + (1-s)\,\phi_1(t)$$

$$H : (X \times I) \times I \to Y$$

$$H(x,t,s) = F(x, s\phi_2(t) + (1-s)\phi_1(t))$$

$$H(x,t,0) = F(x, \phi_1(t)) = G_1$$

$$H(x,t,1) = F(x, \phi_2(t)) = G_2$$

$$H(x,0,s) = F(x, \phi_1(0)) = G_1(x,0)$$

$$H(x,1,s) = F(x, \phi_2(1)) = G_2(x,1)$$

these two follow since

$\phi_1(0) = \phi_2(0)$ &

$\phi_1(1) = \phi_2(1)$

Definition

Let $f: X \to Y$. The **CONSTANT HOMOTOPY** on f, $\mathrm{const}(f): X \times I \to Y$ is defined by

$$\mathrm{Const}(f)(x,t) = f(x) \quad \forall x \in X, t \in I.$$

Proposition

Let $F: X \times I \to Y$ be a homotopy,

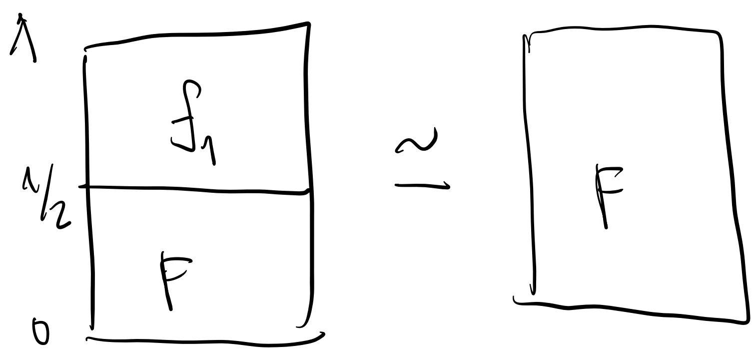
$$f_0 := F\big|_{X \times 0} \qquad f_1 := F\big|_{X \times 1}$$

then

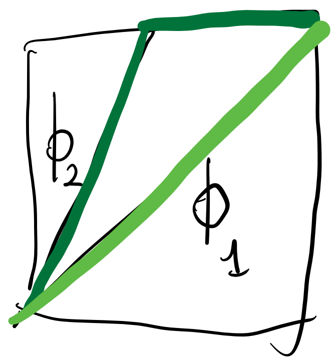
$$F * \mathrm{const}(f_1) \simeq F \text{ rel } (X \times \partial I)$$

$$\mathrm{Const}\, f_2 * F \simeq F \text{ rel } (X \times \partial I)$$

Proof

(1) 

We use reparametrization.

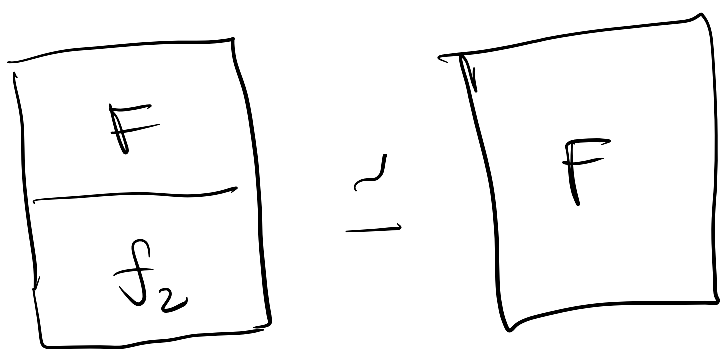


$$\phi_1(t) = t$$

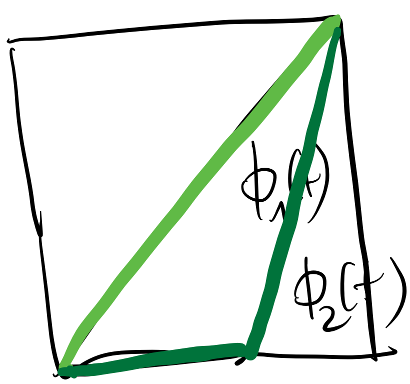
$$\phi_2(t) = \begin{cases} 2t & 0 \le t \le \frac{1}{2} \\ 1 & \frac{1}{2} \le t \le 1 \end{cases}$$

$$\begin{array}{ccc} & & G_2(x,t) \\ & & \| \\ G_1(x,t) = F(x,\phi_1(t)) & \simeq & F(x,\phi_2(t)) \\ \| & & \| \\ F(x,t) & & F * \mathrm{const}_{f_1} \end{array}$$

(2)



We again reparametrize.



$$\phi_1(t) = t$$

$$\phi_2(t) = \begin{cases} 0 & 0 \le t \le \frac{1}{2} \\ 2t-1 & \frac{1}{2} \le t \le 1 \end{cases}$$

THE INVERSE HOMOTOPY

Definition

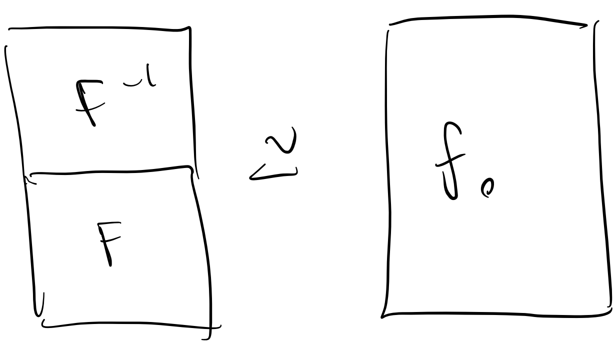
Let $F: X \times I \to Y$ be a homotopy.
then
$F^{-1}: X \times I \to Y$ is defined by

$$F^{-1}(x,t) := F(x, 1-t).$$

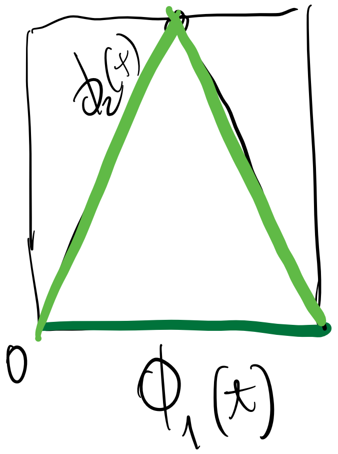
Proposition

$F * F^{-1} \simeq \mathrm{const}(f_0) \ \mathrm{rel}\ (X \times \partial I)$,
where $f_0 := F|_{X \times \{0\}}$.

Proof



We will use the statement about reparametrizations.



$\phi_1(t) = 0$

$$\phi_2(t) = \begin{cases} 2t & 0 \le t \le \frac{1}{2} \\ 2-2t & \frac{1}{2} \le t \le 1 \end{cases}$$

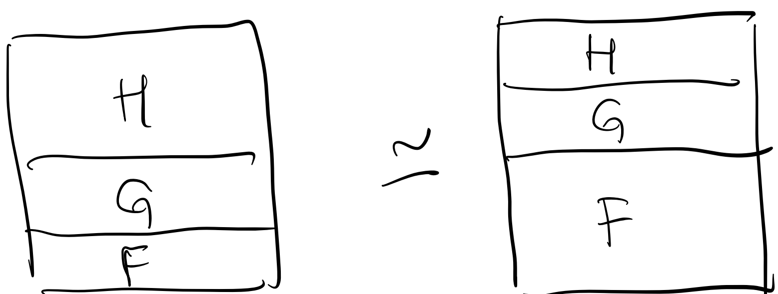
Proposition

Let F, G, H be three homotopies $X \times I \to Y$ s.t. $F * G$ & $G * H$ are defined.

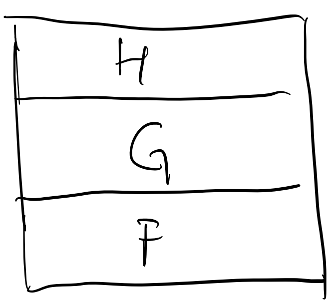
then

$$\big((F * G) * H\big) \simeq F * (G * H) \text{ rel}(X \times \partial I)$$

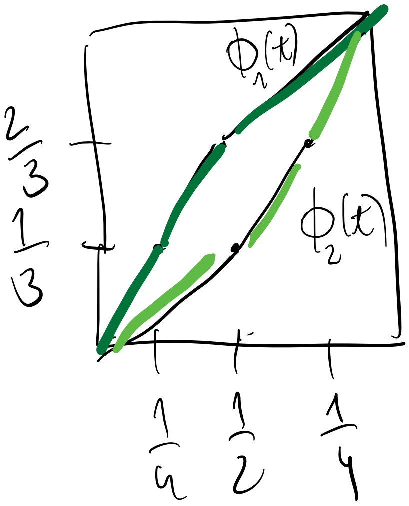
Proof



We again use the reparametrization.



We show that both are homotopy equiv to this one.



Exercise.

Proposition

Let F_1, F_2, G_1, G_2 be homotopies $X \times I \to Y$ with $F_1 \simeq F_2$ rel $(X \times \partial I)$ and $G_1 \simeq G_2$ rel $(X \times \partial I)$ s.t. $F_1(x,1) = G_1(x,0)$ & $F_2(x,1) = G_2(x,0)$ $\forall x \in X$.

Then $F_1 * G_1 \simeq F_2 * G_2$ rel $(X \times \partial I)$.

Proof Exercise.

Proposition

$\sim$ is an equivalence relation on the set of all maps $X \to Y$.