Here is an example of the machinery we developed, a classical result from 1910 due to Brouwer, known as

INVARIANCE OF DIMENSION

If non-empty open sets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$.

Let $x \in U$. By excision

$$H_p(U, U - \{x\}) \cong H_p(\mathbb{R}^m, \mathbb{R}^m - \{x\}).$$

From LES of $(\mathbb{R}^m, \mathbb{R}^m - \{x\})$

$$\cdots \tilde{H}_p(\mathbb{R}^m - \{x\}) \to \tilde{H}_p(\mathbb{R}^m) \to H_p(\mathbb{R}^m, \mathbb{R}^m - \{x\}) \to$$
$$\to \tilde{H}_{p-1}(\mathbb{R}^m - \{x\}) \to \tilde{H}_{p-1}(\mathbb{R}^m) \to$$

we get $\quad H_p(\mathbb{R}^m, \mathbb{R}^m - \{x\}) \cong \tilde{H}_{p-1}(\mathbb{R}^m - \{x\})$

Since $\mathbb{R}^m - \{x\}$ strongly deformation retracts to S^{m-1},

$$H_p(U, U-\{x\}) \cong H_{p-1}(S^{m-1}) = \begin{cases} \mathbb{Z} & p=m \\ 0 & \text{otherwise} \end{cases}$$

Homeomorphism $h: U \to V$ yields a homeomorphism of pairs

$(U, U-\{x\})$ and $(V, V-\{h(x)\})$

and so

$$H_p(U, U-\{x\}) \cong H_p(V, V-\{h(x)\}).$$

Since also

$$H_p(V, V-\{h(x)\}) \cong H_{p-1}(S^{n-1}) = \begin{cases} \mathbb{Z} & p=n \\ 0 & \text{otherwise} \end{cases}$$

it follows that $m=n$.