Number theory I: Problem sheet 1

1. Let K be the splitting field over $\mathbf{Q}$ of the polynomial $X^3 - 2$. Find (with justification) a primitive element of the extension $K/\mathbf{Q}$.

2. Show that if $\mathbb{Q}(\sqrt{n}) = \mathbb{Q}(\sqrt{m})$ for square-free integers n, m not equal to 0 or 1, then $n = m$.

3. (a) Express the polynomial

$$f(X, Y, Z) = X^3Y + Y^3Z + Z^3X + XY^3 + YZ^3 + ZX^3$$

   in terms of the elementary symmetric polynomials.

   (b) Let α, β, γ be the roots of the polynomial $t^3 - 2t + 2$. Determine $f(\alpha, \beta, \gamma)$.

4. Show that if α is algebraic, then so is α^{-1}.

5. (a) Show that $f(t) = t^2 + 4t + 2$ us urreducible over $\mathbb{Q}$.

   (b) Let α be a root of $f(t)$. Find a non-zero monic polynomial in $\mathbf{Q}[t]$ which has $\alpha + \sqrt{2}$ as a root.

6. (a) Let K be a number field, and let $\sigma_1, \dots, \sigma_n$ be the embeddings of K. Let $\alpha_1, \dots, \alpha_n$ be elements of K. Define the matrix $C = (c_{ij})_{1 \le i,j \le n}$ by $c_{ij} = \sigma_i(\alpha_j)$. Show that

$$\Delta[\alpha_1, \dots, \alpha_n] = (\det(C))^2.$$

   (b) Suppose that $\alpha_1, \dots, \alpha_n$ is a $\mathbb{Q}$-basis of K and $\beta_1, \dots, \beta_n \in K$. Define the matrix $D = (d_{ij})$ with $d_{ij} \in \mathbb{Q}$ by

$$\beta_j = \sum_{i=1}^{n} d_{ij}\alpha_i.$$

   Show that

$$\Delta[\beta_1, \dots, \beta_n] = \det(D)^2 \Delta[\alpha_1, \dots, \alpha_n].$$

7. Let $\alpha = \sqrt[3]{2}$, and let $K = \mathbb{Q}(\alpha)$.

   (a) Calculate the norm and trace of the elements $\alpha - 1$ and $\alpha^2 - 1$.

   (b) Calculate $\Delta[1, \alpha, \alpha^2]$.

   (c) Let $\beta = \alpha^2 - 1$. Show that $1, \beta, \beta^2$ is a $\mathbb{Q}$-basis of K, and calculate $\Delta[1, \beta, \beta^2]$.

8. Let α be a root of the polynomial $f(t) = t^3 - t - 1$, and let $K = \mathbb{Q}(\alpha)$. Calculate the norm and trace of the element $\alpha^2 + 1$.