Number theory I: Problem sheet 2

1. Let $d \in \mathbb{Z} \setminus \{0, 1\}$ be square-free and let $K = \mathbb{Q}(\sqrt{d})$. Using the algorithm developed in lectures, show that $1, \tau_d$ is an integral basis of K, where

$$\tau_d = \begin{cases} \sqrt{d} & \text{if } d \not\equiv 1 \pmod 4 \\ \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \pmod 4 \end{cases}$$

2. Let α be a zero of the polynomial $f(t) = t^3 + 2t + 1$, and let $K = \mathbb{Q}(\alpha)$. Show that $O_K = \mathbb{Z}[\alpha]$.

3. Let K be a number field, and let $\theta_1, \ldots, \theta_n$ be an integral basis of K. Let $\alpha_1, \ldots, \alpha_n \in O_K$ be a $\mathbb{Q}$-basis of K. Show that if

$$\Delta[\theta_1, \ldots, \theta_n] = \Delta[\alpha_1, \ldots, \alpha_n], \tag{1}$$

then $\alpha_1, \ldots, \alpha_n$ is also an integral basis of K. Is the converse true?

4. Let $K = \mathbb{Q}(\alpha)$, where $\alpha = \sqrt[3]{3}$. Find an integral basis of K.

5. Let K be a number field of degree n, and let $x_1, \ldots, x_n$ be an integral basis.

   (a) Let $\sigma_1, \ldots \sigma_n$ be the complex embeddings of K, and define

$$P = \sum_{\pi \in A_n} \prod_{i=1}^{n} \sigma_i(x_{\pi(i)}),$$
$$N = \sum_{\pi \notin A_n} \prod_{i=1}^{n} \sigma_i(x_{\pi(i)}),$$

   where A_n is the alternating group on n letters. Show that $P + N$ and PN are in $\mathbb{Z}$.

   (b) Show that $\Delta[x_1, \ldots, x_n] \equiv 0 \text{ or } 1 \pmod 4$.