Number theory I: Problem sheet 3

1. Let $K = \mathbb{Q}(\sqrt{-d})$, where $d > 1$. Show that

$$O_K^\times = \begin{cases} \{\pm 1, \pm\omega, \pm\omega^2 : \omega = e^{\frac{2\pi i}{3}}\} & \text{if } d = 3 \\ \{\pm 1\} & \text{for any other } d > 0 \end{cases}$$

2. Show that for $K = \mathbb{Q}(\sqrt{2})$, the unit group $O_K^\times$ is infinite.

3. Let $\zeta = e^{\frac{2\pi i}{5}}$, and let $K = \mathbb{Q}(\zeta)$.
   (a) Calculate $N(\zeta + 2)$ and $N(\zeta - 2)$.
   (b) Show that $\zeta + 2$ and $\zeta - 2$ have no proper factors (i.e. no factors which aren't units) in $\mathbb{Z}[\zeta]$.

4. Let R be an integral domain. An element $x \in R - R^\times$ is prime if it is nonzero and satisfies

$$x|ab \qquad \Rightarrow \qquad x|a \text{ or } x|b.$$

   (a) Show that any prime is irreducible.
   (b) By considering the factorisation of 6 in $\mathbb{Z}[\sqrt{-5}]$, or otherwise, show that the converse is false.
   (c) Show that $\sqrt{-5}$ is prime in $\mathbb{Z}[\sqrt{-5}]$.

5. (a) Let $f : R \to S$ be a ring homomorphism, let I be an ideal in S. Show that $f^{-1}(I)$ is an ideal of R.
   (b) Is it always true that for R, S rings and $\sigma : R \to S$ a ring homomorphism, the image of an ideal of R under σ is an ideal of S? Give a proof or a counterexample as appropriate.

6. Compute the greatest common divisor in $\mathbb{Z}[i]$ of $4 + 7i$ and $1 + 3i$.