Number theory I: Problem sheet 4

1. Let K be a field, and let $f(t) \in K[t]$ be non-zero. Show that $f(t)$ is irreducible if and only if $K[t]/\langle f(t)\rangle$ is a field.

2. Prove that a subset $\mathfrak{b}$ of a number field K is a fractional ideal if and only if the following conditions are satisfied: (a) if $x, y \in \mathfrak{b}$, then $x + y \in \mathfrak{b}$, (b) $\mathfrak{b}O_K \subset \mathfrak{b}$, and (c) there exists $x \in O_K$ such that $x\mathfrak{b} \subseteq O_K$.

3. Let $K = \mathbb{Q}(\sqrt{-3})$, and let $\mathfrak{a} = \left\langle 2, \frac{1-\sqrt{-3}}{2} \right\rangle$. Determine $\mathfrak{a}^{-1}$.

4. Let $K = \mathbb{Q}(\sqrt{-5})$, and consider the ideals $\mathfrak{p}_1 = \langle 2, 1 + \sqrt{-5}\rangle$ and $\mathfrak{p}_2 = \langle 3, 1 - \sqrt{-5}\rangle$ in O_K.
   (a) Show that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are maximal, and compute $|O_K/\mathfrak{p}_i|$ for $i = 1, 2$.
   (b) Show that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ can't be principal ideals.
   (c) Show that $\mathfrak{p}_1\mathfrak{p}_2$ is principal, and find a generator.

5. Determine all fractional ideals in $\mathbb{Z}[i]$.

6. Let $\mathfrak{a}$ be a non-zero ideal in O_K. Show that $\mathfrak{a} | \langle N(\mathfrak{a})\rangle$.