Number theory I: Problem sheet 5

1. Define the ideals

$$\mathfrak{p} = \langle 2, 1+\sqrt{-5} \rangle$$
$$\mathfrak{q} = \langle 3, 1+\sqrt{-5} \rangle$$

   in $\mathbb{Z}[\sqrt{-5}]$.

   (a) Calculate $\mathfrak{p}^2$ and $\mathfrak{p}\mathfrak{q}$.

   (b) Are $\mathfrak{p}$ and $\mathfrak{q}$ maximal? Justify your answer.

   (c) Determine a $\mathbb{Z}$-basis for each of the ideals.

2. (a) Find all the ideals $\mathfrak{a}$ in $\mathbb{Z}[\sqrt{-5}]$ such that $6 \in \mathfrak{a}$.

   (b) Find all the ideals in $\mathbb{Z}[\sqrt{5}]$ of norm 18.

   (c) Let $L = \mathbb{Q}(\sqrt{-3})$. Show that there is unique ideal in O_L of norm 12.

3. Consider the ideal $\mathfrak{a} = \langle 5 - 2\sqrt{-5} \rangle$ in $\mathbb{Z}[\sqrt{-5}]$. Factorise $\mathfrak{a}$ into maximal ideals.

4. Determine all ideals in $O_K = \mathbb{Z}[\sqrt{6}]$ of norm 24.

5. Let α be a root of $t^3 + 2t + 2$, and let $K = \mathbb{Q}(\alpha)$.

   (a) Show that $\{1, \alpha, \alpha^2\}$ is an integral basis of K.

   (b) Factorise p into maximal ideals in O_K for $p = 5, 7$.

   (c) Calculate $N(3-\alpha)$ and hence factorise $\langle 3 - \alpha \rangle$ into maximal ideals.

   (d) Show that $5 - \alpha$ is irreducible in O_K.