Number theory I: Problem sheet 8

1. Let $K = \mathbb{Q}(\sqrt[3]{2})$
   (a) Compute an integral basis of K.
   (b) Compute the ideal class group of K.
2. Let $p \geq 3$ be an odd prime, and let $\zeta = e^{2\pi i/p}$ and $F = \mathbb{Q}(\zeta)$.
   (a) Show that $\langle 1 - \zeta \rangle$ is prime.
   (b) Show that there exists $u \in O_F^\times$ such that $p = u(1 - \zeta)^{p-1}$.
   (c) Show that the only roots of unity in O_F are of the form $\pm\zeta^s$ for some $s \in \mathbb{Z}$.
   (d) Show that for r, s integers, both of them coprime to p, we have
   $$\frac{\zeta^r - 1}{\zeta^s - 1} \in O_F^\times$$
3. Let K be a number field, and let $N \geq 1$. Then there are only finitely many $\alpha \in O_K$ such that all conjugates of α have complex absolute value $\leq N$.