Number theory I: Problem sheet 9

1. Let α be an element in a number field K, and suppose that $|N(\alpha)| = 1$. Does this imply that $\alpha \in O_K^\times$? Give a proof or counterexample as appropriate.

2. Let $K = Q(\sqrt{d})$ be a real quadratic field, and write the fundamental unit $u = a + b\sqrt{d}$. Show that $b > 0$.

3. Calculate the fundamental unit of $Q(\sqrt{d})$ for $d = 3, 5, 7, 11, 13, 15$. Hence determine all solutions of Pell's equation $x^2 - dy^2 = \pm 1$.

4. Write $u = 80 + 9\sqrt{79}$ for the fundamental unit in $\mathbb{Z}[\sqrt{79}]$. *(You may assume without proof that u is the fundamental unit.)*

   (a) Show that if there exists an element $z \in \mathbb{Z}[\sqrt{79}]$ with norm -3, then there is one which satisfies $1 < z < u$.

   (b) Show that if we write that element as $z = a + b\sqrt{79}$, then

   $$1 + \frac{3}{u} < 2b\sqrt{79} < 3 + u.$$