Problem sheet 1 Solutions

Problem 1

The splitting field K over $\mathbb{Q}$ of the polynomial $x^3 - 2$ is $\mathbb{Q}(\alpha, \omega)$, where $\alpha = 2^{\frac{1}{3}}$ and $\omega = e^{\frac{2\pi i}{3}} \in \mathbb{C}$. We will show that $\beta = \alpha + \omega$ is a primitive element of $K/\mathbb{Q}$. Since the degree of $K/\mathbb{Q}$ is 6, it suffices to check that $1, \beta, \beta^2, \beta^3$ are $\mathbb{Q}$-linearly independent. By a straightforward computation with a basis $\{1, \alpha, \alpha^2, \omega, \alpha\omega, \alpha^2\omega\}$ of $K/\mathbb{Q}$, we have

$$\begin{aligned} 1 &= 1, \\ \beta &= \alpha + \omega, \\ \beta^2 &= -1 + \alpha^2 - \omega + 2\alpha\omega, \\ \beta^3 &= 3 - 3\alpha - 3\alpha\omega + 3\alpha^2\omega, \end{aligned}$$

hence $1, \beta, \beta^2, \beta^3$ are $\mathbb{Q}$-linearly independent.

Problem 2

If $\mathbb{Q}(\sqrt{n}) = \mathbb{Q}(\sqrt{m})$, then there exist $a, b \in \mathbb{Q}$ such that $\sqrt{n} = a + b\sqrt{m}$.

Case 1: $a \neq 0$ and $b \neq 0$. By squaring the equation, we get $\sqrt{m} = \frac{n - a^2 - mb^2}{2ab}$. This is impossible since m is square-free.

Case 2: $b = 0$. In this case we have $\sqrt{n} = a$ but this is also impossible.

Case 3: $a = 0$ and $b \neq 0$. In this case we have $\sqrt{n} = b\sqrt{m}$. Let us write $b = \frac{p}{q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Then $q^2 n = p^2 m$. Since m and n are square-free, we conclude that $p^2 = q^2 = 1$ and $m = n$.

Problem 3

(a) We denote $s_1 = X + Y + Z$, $s_2 = XY + YZ + ZX$, and $s_3 = XYZ$. Then

$$\begin{aligned} f(X, Y, Z) &= (XY + YZ + ZX)(X^2 + Y^2 + Z^2) - XYZ(X + Y + Z) \\ &= s_2(s_1^2 - 2s_2) - s_1 s_3 = s_1^2 s_2 - 2s_2^2 - s_1 s_3. \end{aligned}$$

(b) By Vieta's formulas, we have $s_1 = 0$, $s_2 = -2$, and $s_3 = -2$. It follows that

$$f(\alpha, \beta, \gamma) = s_1^2 s_2 - 2s_2^2 - s_1 s_3 = -8.$$

Problem 4

We may assume $\alpha \neq 0$. If α is algebraic, then there exists $a_0, \cdots, a_{n-1} \in \mathbb{Q}$ such that $\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0$. Let $0 \leq i \leq n-1$ be the smallest integer such that $a_i \neq 0$. Multiplying the equation by α^{-n}, we get $1 + a_{n-1}\alpha^{-1} + \cdots + a_i\alpha^{-(n-i)} = 0$, hence α^{-1} is algebraic.

Problem 5

(a) The polynomial $f(t) = t^2 + 4t + 2$ has $-2+\sqrt{2}$ and $-2-\sqrt{2}$ as roots over $\mathbb{R}$ but the roots are not in $\mathbb{Q}$. Thus f is irreducible over $\mathbb{Q}$.

(b) As constructed in Remark 2.22 using symmetric polynomials, $f(t-\sqrt{2})f(t+\sqrt{2})$ is a nonzero monic polynomial and has $\alpha+\sqrt{2}$ as a root. Note that

$$f(t-\sqrt{2})f(t+\sqrt{2}) = t^4 + 8t^3 + 16t^2 - 16.$$

Problem 6

(a) It follows from the following computation:

$$\begin{aligned}
\Delta[\alpha_1, \cdots, \alpha_n] &= \det\big(\operatorname{Tr}(\alpha_i\alpha_j)\big)_{1\leq i,j\leq n} \\
&= \det\Big(\sum_{k=1}^{n} \sigma_k(\alpha_i\alpha_j)\Big)_{1\leq i,j\leq n} \\
&= \det\Big(\sum_{k=1}^{n} \sigma_k(\alpha_i)\sigma_k(\alpha_j)\Big)_{1\leq i,j\leq n} \\
&= \det\big(\sigma_k(\alpha_i)\big)_{1\leq i,k\leq n} \det\big(\sigma_k(\alpha_j)\big)_{1\leq j,k\leq n} \\
&= \det\big(\sigma_k(\alpha_i)\big)^2_{1\leq i,k\leq n} = \big(\det(C)\big)^2.
\end{aligned}$$

(b) Using (a), we obtain

$$\begin{aligned}
\Delta[\beta_1, \cdots, \beta_n] &= \det\big(\sigma_k(\beta_j)\big)^2_{1\leq j,k\leq n} \\
&= \det\Big(\sum_{i=1}^{n} d_{ij}\sigma_k(\alpha_i)\Big)^2_{1\leq j,k\leq n} \\
&= \det\big(d_{ij}\big)^2_{1\leq i,j\leq n} \det\big(\sigma_k(\alpha_i)\big)^2_{1\leq i,k\leq n} \\
&= \det\big(D\big)^2 \Delta[\alpha_1, \cdots, \alpha_n].
\end{aligned}$$

Problem 7

(a) Let $\sigma_1, \sigma_2, \sigma_3$ be the complex embeddings of K and denote $\theta_i = \sigma_i(\alpha)$ for $i = 1, 2, 3$. We also denote $s_1 = \theta_1 + \theta_2 + \theta_3$, $s_2 = \theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1$, and $s_3 = \theta_1\theta_2\theta_3$. Note that the minimal polynomial of α is $t^3 - 2$. By Vieta's formulas we have $s_1 = s_2 = 0$ and $s_3 = 2$. It follows that

$$\begin{aligned} N_{K/\mathbb{Q}}(\alpha - 1) &= \prod_{i=1}^{3} \sigma_i(\alpha - 1) = \prod_{i=1}^{3} (\sigma_i(\alpha) - 1) = (\theta_1 - 1)(\theta_2 - 1)(\theta_3 - 1) \\ &= s_3 - s_2 + s_1 - 1 = 1, \end{aligned}$$

$$\begin{aligned} N_{K/\mathbb{Q}}(\alpha^2 - 1) &= \prod_{i=1}^{3} \sigma_i(\alpha^2 - 1) = \prod_{i=1}^{3} (\sigma_i(\alpha)^2 - 1) = (\theta_1^2 - 1)(\theta_2^2 - 1)(\theta_3^2 - 1) \\ &= \theta_1^2\theta_2^2\theta_3^2 - (\theta_1^2\theta_2^2 + \theta_2^2\theta_3^2 + \theta_3^2\theta_1^2) + (\theta_1^2 + \theta_2^2 + \theta_3^2) - 1 \\ &= s_3^2 - (s_2^2 - 2s_1s_3) + (s_1^2 - 2s_2) - 1 = 3, \end{aligned}$$

$$\begin{aligned} \mathrm{Tr}_{K/\mathbb{Q}}(\alpha - 1) &= \sum_{i=1}^{3} \sigma_i(\alpha - 1) = \sum_{i=1}^{3} (\sigma_i(\alpha) - 1) \\ &= \theta_1 + \theta_2 + \theta_3 - 3 = s_1 - 3 = -3, \end{aligned}$$

$$\begin{aligned} \mathrm{Tr}_{K/\mathbb{Q}}(\alpha^2 - 1) &= \sum_{i=1}^{3} \sigma_i(\alpha^2 - 1) = \sum_{i=1}^{3} (\sigma_i(\alpha)^2 - 1) \\ &= \theta_1^2 + \theta_2^2 + \theta_3^2 - 3 = s_1^2 - 2s_2 - 3 = -3. \end{aligned}$$

(b) We calculate the following determinant of the matrix:

$$\Delta[1, \alpha, \alpha^2] = \det \begin{pmatrix} \mathrm{Tr}(1) & \mathrm{Tr}(\alpha) & \mathrm{Tr}(\alpha^2) \\ \mathrm{Tr}(\alpha) & \mathrm{Tr}(\alpha^2) & \mathrm{Tr}(\alpha^3) \\ \mathrm{Tr}(\alpha^2) & \mathrm{Tr}(\alpha^3) & \mathrm{Tr}(\alpha^4) \end{pmatrix}.$$

The traces of the powers of α are

$$\mathrm{Tr}(1) = \sum_{i=1}^{3} \sigma_i(1) = 3,$$

$$\mathrm{Tr}(\alpha) = \sum_{i=1}^{3} \sigma_i(\alpha) = s_1 = 0,$$

$$\mathrm{Tr}(\alpha^2) = \sum_{i=1}^{3} \sigma_i(\alpha)^2 = s_1^2 - 2s_2 = 0,$$

$$\mathrm{Tr}(\alpha^3) = \mathrm{Tr}(2) = \sum_{i=1}^{3} \sigma_i(2) = 6,$$

$$\mathrm{Tr}(\alpha^4) = \mathrm{Tr}(2\alpha) = 0.$$

It follows that

$$\Delta[1, \alpha, \alpha^2] = \det \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 6 \\ 0 & 6 & 0 \end{pmatrix} = -108.$$

Alternatively, one may also apply Proposition 2.39 and compute

$$\Delta[1, \alpha, \alpha^2] = (-1)^{\frac{3\times 2}{2}} N_{K/\mathbb{Q}}(3\alpha^2) = -27 N_{K/\mathbb{Q}}(\alpha)^2 = -108.$$

(c) Note that

$$\begin{pmatrix} 1 \\ \beta \\ \beta^2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 + \alpha^2 \\ 1 - 2\alpha - 2\alpha^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & -2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \\ \alpha^2 \end{pmatrix}.$$

Using (b) of Problem 6, we get

$$\Delta[1, \beta, \beta^2] = \det \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & -2 & -2 \end{pmatrix}^2 \Delta[1, \alpha, \alpha^2] = -432.$$

Problem 8

(a) Let $\sigma_1, \sigma_2, \sigma_3$ be the complex embeddings of K and denote $\theta_i = \sigma_i(\alpha)$ for $i = 1, 2, 3$. We also denote $s_1 = \theta_1 + \theta_2 + \theta_3$, $s_2 = \theta_1\theta_2 + \theta_2\theta_3 + \theta_3\theta_1$, and $s_3 = \theta_1\theta_2\theta_3$. Then the norm and the trace are expressed by

$$\begin{aligned} N_{K/\mathbb{Q}}(\alpha^2 + 1) &= \prod_{i=1}^{3} \sigma_i(\alpha^2 + 1) = \prod_{i=1}^{3} (\sigma_i(\alpha)^2 + 1) \\ &= (\theta_1^2 + 1)(\theta_2^2 + 1)(\theta_3^2 + 1) \\ &= 1 + (\theta_1^2 + \theta_2^2 + \theta_3^2) + (\theta_1^2\theta_2^2 + \theta_2^2\theta_3^2 + \theta_3^2\theta_1^2) + \theta_1^2\theta_2^2\theta_3^2 \\ &= 1 + (s_1^2 - 2s_2) + (s_2^2 - 2s_1s_3) + s_3^2, \end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}_{K/\mathbb{Q}}(\alpha^2+1) &= \sum_{i=1}^{3} \sigma_i(\alpha^2+1) = \sum_{i=1}^{3} (\sigma_i(\alpha)^2+1) \\
&= \theta_1^2+\theta_2^2+\theta_3^2+3 \\
&= s_1^2 - 2s_2 + 3.
\end{aligned}$$

On the other hand, we have $s_1 = 0$, $s_2 = -1$, and $s_3 = 1$ from Vieta's formulas. Thus $N_{K/\mathbb{Q}}(\alpha^2+1) = 5$ and $\mathrm{Tr}_{K/\mathbb{Q}}(\alpha^2+1) = 5$.