Problem sheet 10 Solutions

Problem 1

There exist x, y in the positive integers satisfying $x^2 - 15y^2 = n$ if any only if there exists an element $x + y\sqrt{15}$ of norm n. In Problem 3 of the previous problem sheet we saw that $4 + \sqrt{15}$ is the fundamental unit of norm 1, and hence there is no unit of norm -1.

(1) $n = \pm 4$.

We deduce from Dedekind's criterion that $\langle 2 \rangle = \mathfrak{p}_2^2$, where $\mathfrak{p}_2 = \langle 2, 1+\sqrt{15} \rangle$. It follows that $\langle 2 \rangle$ is the unique ideal of norm 4, hence the solutions for $n = 4$ are just the twice of the solutions for $n = 1$, i.e. $2(4 + \sqrt{15})^m$ for $m \in \mathbb{Z}$. Since there is no unit of norm -1, there is no element of norm -4, hence the equation for $n = -4$ is not solvable.

(2) $n = \pm 7$.

We deduce from Dedekind's criterion that $\langle 7 \rangle = \mathfrak{p}_7\mathfrak{p}_7'$, where $\mathfrak{p}_7 = \langle 7, 1+\sqrt{15} \rangle$ and $\mathfrak{p}_7' = \langle 7, 1 - \sqrt{15} \rangle$.

One can check that the ideals $\langle 7, 1 + \sqrt{15} \rangle$ and $\langle 7, 1 - \sqrt{15} \rangle$ are not principal. Indeed, as the norm of $\langle 1 + \sqrt{15} \rangle$ and $\langle 1 - \sqrt{15} \rangle$ are 14, their factorization must be either $\mathfrak{p}_2\mathfrak{p}_7$ or $\mathfrak{p}_2\mathfrak{p}_7'$. In any case, $\mathfrak{p}_7$ and $\mathfrak{p}_7'$ are not principal since$\mathfrak{p}_2$ is not principal.

It follows that there is no element of norm ± 7, hence the equations for $n = \pm 7$ are not solvable.

(3) $n = \pm 11$.

We deduce from Dedekind's criterion that $\langle 11 \rangle = \mathfrak{p}_{11}\mathfrak{p}_{11}'$, where $\mathfrak{p}_{11} = \langle 11, 2+\sqrt{15} \rangle = \langle 2 + \sqrt{15} \rangle$, $\mathfrak{p}_{11}' = \langle 11, 2 - \sqrt{15} \rangle = \langle 2 - \sqrt{15} \rangle$.

As $N(2 + \sqrt{15}) = N(2 - \sqrt{15}) = -11$, the solutions for $n = -11$ are $(2 + \sqrt{15})(4 + \sqrt{15})^m$ and $(2 - \sqrt{15})(4 + \sqrt{15})^m$ for $n \in \mathbb{Z}$. There is no element of norm 11, hence the equation for $n = 11$ is not solvable.

Problem 2

In the lecture we saw that $3 + \sqrt{10}$ is the fundamental unit of norm -1.

(1) $n = 7$.

We deduce from Dedekind's criterion that $\langle 7 \rangle = \mathfrak{p}_7$ is maximal. It follows that there is no element of norm 7, hence there is no solution to $x^2 - 10y^2 = 7$.

(2) $n = 8$.

We deduce from Dedekind's criterion that $\langle 2 \rangle = \mathfrak{p}_2^2$, where $\mathfrak{p}_2 = \langle \sqrt{10}, 2 \rangle$. It follows that $\mathfrak{p}_2^3$ is the unique ideal of norm 8. Since $\mathfrak{p}_2$ is not principal and $\mathfrak{p}_2^3 = 2\mathfrak{p}_2$, there is no solution to $x^2 - 10y^2 = 8$.

Problem 3

Note that for any $j = 1, \cdots, r+s-1$ we have

$$\log|\sigma_1(\epsilon_j)| + \cdots + \log|\sigma_r(\epsilon_j)| + 2\log|\sigma_{r+1}(\epsilon_j)| + 2\log|\sigma_{r+s}(\epsilon_j)| = 0.$$

Thus it is sufficient to prove the following claim.

Claim. Let $M = (a_{ij})_{1\le i\le n, 1\le j\le n-1}$ and suppose that $\sum_{i=1}^{n} a_{ij} = 0$. Let M_i be the matrix obtained from M by deleting the ith column. Then $|\det(M_i)|$ is independent of i.

*Proof of **Claim.*** Without loss of generality let us show $|\det(M_1)| = |\det(M_n)|$. Let v_i be the ith column of M. It follows from $\sum_{i=1}^{n} a_{ij} = 0$ that $\sum_{i=1}^{n} v_i = 0$. By the properties of determinant we have

$$\begin{aligned}|\det(M_1)| = |\det\begin{pmatrix}v_2 & \cdots & v_n\end{pmatrix}| &= |\det\begin{pmatrix}v_2 & \cdots & v_{n-1} & -v_1 - \cdots - v_{n-1}\end{pmatrix}| \\ &= |\det\begin{pmatrix}v_2 & \cdots & v_{n-1} & -v_1\end{pmatrix}| \\ &= |\det\begin{pmatrix}v_1 & v_2 & \cdots & v_{n-1}\end{pmatrix}| = |\det(M_n)|.\end{aligned}$$

□

Problem 4

For $A \in \mathrm{Cl}(K)$ and $\mathfrak{a} \in A^{-1}$ we clearly have $[\mathfrak{a}^{-1}] = A$. For $\mathfrak{a} \in A^{-1}$ and an integral ideal $\mathfrak{c}$ in A, the ideal $\mathfrak{ac}$ is in $[\langle 1\rangle]$, hence principal. Thus the map $\mathfrak{c} \mapsto \mathfrak{ac}$ sends an integral ideal in A to a principal ideal divisible by $\mathfrak{a}$.

We first show that the map $\mathfrak{c} \mapsto \mathfrak{ac}$ is injective. Let $\mathfrak{c}_1$ and $\mathfrak{c}_2$ be integral ideals in A and suppose that $\mathfrak{ac}_1 = \mathfrak{ac}_2$. It directly implies $\mathfrak{c}_1 = \mathfrak{a}^{-1}(\mathfrak{ac}_1) = \mathfrak{a}^{-1}(\mathfrak{ac}_2) = \mathfrak{c}_2$.

We next show that the map is surjective. Let $\mathfrak{b}$ be a principal ideal divisible by $\mathfrak{a}$. Then there exists an integral ideal $\mathfrak{c}$ such that $\mathfrak{b} = \mathfrak{ac}$. Since $[\mathfrak{a}][\mathfrak{c}] = [\mathfrak{b}] = [\langle 1\rangle]$, we have $[\mathfrak{c}] = A$, hence $\mathfrak{c}$ is an integral ideal in A.

Problem 5

Suppose that $x \in K_\infty$ satisfies
(1) $N(x) \neq 0$,
(2) $0 \le \gamma_i < 1$ for all $1 \le i \le r+s-1$, where

$$\mathcal{L}(x) = \gamma_0\lambda + \gamma_1 u_1 + \cdots + \gamma_{r+s-1}u_{r+s-1},$$

(3) $0 \leq \arg(x_1) < \frac{2\pi}{|W_K|}$.

Let $\alpha > 0$. Then we have

(1) $N(\alpha x) = \alpha^{r+2s} N(x) \neq 0$,

(2)

$$\mathcal{L}(\alpha x) = \begin{pmatrix} \log|\sigma_1(\alpha x)| \\ \ddots \\ \log|\sigma_r(\alpha x)| \\ 2\log|\sigma_{r+1}(\alpha x)| \\ \ddots \\ 2\log|\sigma_{r+s}(\alpha x)| \end{pmatrix} = \begin{pmatrix} \log|\sigma_1(x)| \\ \ddots \\ \log|\sigma_r(x)| \\ 2\log|\sigma_{r+1}(x)| \\ \ddots \\ 2\log|\sigma_{r+s}(x)| \end{pmatrix} + \begin{pmatrix} \log\alpha \\ \ddots \\ \log\alpha \\ 2\log\alpha \\ \ddots \\ 2\log\alpha \end{pmatrix} = \mathcal{L}(x) + \lambda \log\alpha,$$

hence $\mathcal{L}(\alpha x) = (\gamma_0 + \log\alpha)\lambda + \gamma_1 u_1 + \cdots + \gamma_{r+s-1} u_{r+s-1}$,

(3) $\arg(\alpha x_1) = \arg(x_1)$.

It follows that the set $X \subset K_\infty$ defined in Definition 10.18 is a cone.