Problem sheet 9 Solutions

Problem 1

Let $K = \mathbb{Q}(i)$ and $\alpha = \frac{3}{5} + \frac{4}{5}i$. Then $N(\alpha) = \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = 1$ but α is not an algebraic integer as the minimal polynomial of α is $5x^20 - 6x + 5$.

Problem 2

Let $u = a + b\sqrt{d}$ and $|N(u)| = 1$. Note that the fundamental unit is the largest value from $\{u, u^{-1}, -u, -u^{-1}\} = \{a + b\sqrt{d}, a - b\sqrt{d}, -a + b\sqrt{d}, -a - b\sqrt{d}\}$. It follows that if $u = a + b\sqrt{d}$ is the fundamental unit, then $a, b > 0$.

Problem 3

Recall that the fundamental unit is given by $a + b\sqrt{d}$, where (a, b) is the solution to $a^2 - db^2 = \pm 1$ with (a, b) positive integers (if $d \not\equiv 1 (\text{mod } 4)$) or half-integers (if $d \equiv 1 (\text{mod } 4)$) having the smallest possible value of a.
1) $d = 3$.
The smallest solution is $(a, b) = (2, 1)$, hence the fundamental unit is $u = 2 + \sqrt{3}$. It follows that all integral solutions of $x^2 - dy^2 = \pm 1$ are (a_n, b_n) with $(2 + \sqrt{3})^n = a_n + b_n\sqrt{3}$, where $n \in \mathbb{Z}$.
2) $d = 5$.
The smallest solution is $(a, b) = (\frac{1}{2}, \frac{1}{2})$, hence the fundamental unit is $u = \frac{1}{2} + \frac{\sqrt{5}}{2}$. It follows that all half-integral solutions of $x^2 - dy^2 = \pm 1$ are (a_n, b_n) with $(\frac{1+\sqrt{5}}{2})^n = a_n + b_n\sqrt{5}$, where $n \in \mathbb{Z}$. The smallest positive integer n such that a_n and b_n are integer is $n = 3$ as $(a_3, b_3) = (2, 1)$. One can also check that a_n, b_n are integers if and only if n is divisible by 3, using the recurrence relations

$$a_{n+1} = \frac{1}{2}a_n + \frac{5}{2}b_n, \quad b_{n+1} = \frac{1}{2}a_n + \frac{1}{2}b_n.$$

Thus, all integral solutions of $x^2 - dy^2 = \pm 1$ are (a_{3n}, b_{3n}), where $n \in \mathbb{Z}$.
3) $d = 7$.
The smallest solution is $(a, b) = (8, 3)$, hence the fundamental unit is $u = 8 + 3\sqrt{7}$. It follows that all integral solutions of $x^2 - dy^2 = \pm 1$ are (a_n, b_n) with $(8 + 3\sqrt{7})^n = a_n + b_n\sqrt{7}$, where $n \in \mathbb{Z}$.
4) $d = 11$.
The smallest solution is $(a, b) = (10, 3)$, hence the fundamental unit is $u = 10 + 3\sqrt{11}$. It follows that all integral solutions of $x^2 - dy^2 = \pm 1$ are (a_n, b_n) with $(10 + 3\sqrt{11})^n = a_n + b_n\sqrt{11}$, where $n \in \mathbb{Z}$.

5) $d = 13$.
The smallest solution is $(a,b) = (\frac{3}{2}, \frac{1}{2})$, hence the fundamental unit is $u = \frac{3}{2} + \frac{\sqrt{13}}{2}$. It follows that all half-integral solutions of $x^2 - dy^2 = \pm 1$ are (a_n, b_n) with $(\frac{3+\sqrt{13}}{2})^n = a_n + b_n\sqrt{13}$, where $n \in \mathbb{Z}$. The smallest positive integer n such that a_n and b_n are integer is $n = 3$ as $(a_3, b_3) = (18, 5)$. One can also easily check that a_n, b_n are integers if and only if n is divisible by 3, using the recurrence relations

$$a_{n+1} = \frac{3}{2}a_n + \frac{13}{2}b_n, \quad b_{n+1} = \frac{1}{2}a_n + \frac{3}{2}b_n.$$

Thus, all integral solutions of $x^2 - dy^2 = \pm 1$ are (a_{3n}, b_{3n}), where $n \in \mathbb{Z}$.
4) $d = 15$.
The smallest solution is $(a,b) = (4,1)$, hence the fundamental unit is $u = 4 + \sqrt{15}$. It follows that all integral solutions of $x^2 - dy^2 = \pm 1$ are (a_n, b_n) with $(4 + \sqrt{15})^n = a_n + b_n\sqrt{15}$, where $n \in \mathbb{Z}$.

Problem 4

(1) Note that $N(u) = 1$. Let $z_0 \in \mathbb{Z}[\sqrt{79}]$ be an element such that $N(z_0) = -3$. We may assume $z_0 > 0$. Observe that for any $k \in \mathbb{Z}$, $u^k z_0$ are also elements of norm -3. It follows that there exists k such that $1 \leq u^k z_0 < u$. Clearly $u^k z_0$ is not 1, so $z = u^k z_0$ satisfies $1 < z < u$.

(2) By (1) we have $1 < a + b\sqrt{79} < u$. Since

$$a - b\sqrt{79} = \frac{a^2 - 79b^2}{a + b\sqrt{79}} = \frac{N(z)}{a + b\sqrt{79}} = \frac{-3}{a + b\sqrt{79}},$$

we have $\frac{3}{u} < b\sqrt{79} - a < 3$. Combining two inequalities, we obtain

$$1 + \frac{3}{u} < 2b\sqrt{79} < 3 + u.$$