Exercise sheet 1

1. Let G be a Hausdorff topological group, and $H < G$ a subgroup.
   (a) Show that if H is closed and has finite index in G, then H is open.
   (b) Show that if H is abelian, then so is its closure $\overline{H}$.
   (c) Recall that a group G is *solvable* if there exists a chain $G \rhd G_1 \rhd G_2 \dots G_n \rhd (e) = G_{n+1}$ with $G_{i+1} \lhd G_i$ and G_i/G_{i+1} abelian for all $0 \leqslant i \leqslant n$. Show that if G is solvable then one can also find a chain of subgroups G_i as above with G_i closed in G.

2. (a) Find an injection $O(1,1) \hookrightarrow O(p,q)$ for $p,q \geqslant 1$.
   (b) Show that the topological group $O(p,q)$ for $p,q \geqslant 1$ is not compact.
   (c) Show that $O(1,1)$ has four connected components.

3. (a) Let X be a *compact* Hausdorff space. Show that $(\mathrm{Homeo}(X), \circ)$ is a topological group when endowed with the compact-open topology.
   (b) The objective of this exercise is to show that $(\mathrm{Homeo}(X), \circ)$ will not necessarily be a topological group if X is only locally compact.
   Consider the "middle thirds" Cantor set
   $$C = \left\{ \sum_{n=1}^{\infty} \varepsilon_n 3^{-n} : \varepsilon_n \in \{0,2\} \text{ for each } n \in \mathbb{N} \right\} \subset [0,1]$$
   in the unit interval. We define the sets $U_n = C \cap [0, 3^{-n}]$ and $V_n = C \cap [1 - 3^{-n}, 1]$. Further we construct a sequence of homeomorphisms $h_n \in \mathrm{Homeo}(C)$ as follows:
   - $h_n(x) = x$ for all $x \in C \smallsetminus (U_n \cup V_n)$,
   - $h_n(0) = 0$,
   - $h_n(U_{n+1}) = U_n$,
   - $h_n(U_n \smallsetminus U_{n+1}) = V_{n+1}$,
   - $h_n(V_n) = V_n \smallsetminus V_{n+1}$.

   These restrict to homeomorphisms $h_n|_X$ on $X := C \smallsetminus \{0\}$.
   Show that the sequence $(h_n|_X)_{n \in \mathbb{N}} \subset \mathrm{Homeo}(X)$ converges to the identity on X but the sequence $((h_n|_X)^{-1})_{n \in \mathbb{N}} \subset \mathrm{Homeo}(X)$ of their inverses does not!
   Remark: However, if X is locally compact and *locally connected* then $\mathrm{Homeo}(X)$ is a topological group.

4. Show that if M is a manifold of dimension at least one, then $\mathrm{Homeo}(M)$ is not locally compact.

5. Let (X, d) be a proper metric space. Recall that the isometry group of X is defined as

$$\mathrm{Iso}(X) = \{f \in \mathrm{Homeo}(X) : d(f(x), f(y)) = d(x, y) \text{ for all } x, y \in X\}.$$

   (a) Show that Iso(X) $\subset \mathrm{Homeo}(X)$ is locally compact with respect to the compact-open topology.

   (b) Show that if additionally X is compact, then $\mathrm{Iso}(X)$ is compact.

   Hint: Use the fact that the compact-open topology is induced by the metric of uniform-convergence and apply Arzelà–Ascoli's theorem.