Exercise sheet 3

1. Let $H < G$ be a subgroup of a topological group. Show that the action $G \times G/H \to G/H$ is continuous.

2. In the setting of Example 2.44 (1), show that if $1 \leqslant k \leqslant n-1$ then $\mathrm{SO}(n,\mathbb{R})$ acts transitively on GO_k.

3. In the setting of Example 2.44 (2), let $P := P_{(1,\dots,n-1)}$ be the subgroup of $\mathrm{GL}(n,\mathbb{R})$ of upper triangular matrices. Show that $\mathrm{GL}(n,\mathbb{R})/P$ is compact and deduce that $\mathrm{GL}(n,\mathbb{R})/P_d$ is compact as well.

4. $\mathrm{SL}(2,\mathbb{R})$ acts transitively on $\mathbb{R}^2 \smallsetminus \{0\}$ and the orbit map
$$\mathrm{SL}(2,\mathbb{R})/N \to \mathbb{R}^2 \smallsetminus \{0\}, \quad gN \mapsto g(e_1),$$
where $N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}$ is an $\mathrm{SL}(2,\mathbb{R})$-equivariant homeomorphism. Using this show that there is an $\mathrm{SL}(2,\mathbb{R})$-invariant regular Borel measure on $\mathrm{SL}(2,\mathbb{R})/N$.

5. $\mathrm{SL}(2,\mathbb{R})$ acts transitively on the projective line $\mathbb{P}^1(\mathbb{R})$ and let
$$B = \mathrm{Stab}(\mathbb{R}e_1) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^\times, b \in \mathbb{R} \right\}.$$
Show that on $\mathrm{SL}(2,\mathbb{R})/B$ there is no $\mathrm{SL}(2,\mathbb{R})$-invariant regular Borel measure.