Exercise sheet 4

1. Let T_ℓ, $\ell \geqslant 3$, be the ℓ-regular tree together with the combinatorial distance d. Show that $\mathrm{Iso}(T_\ell, d)$ is not a Lie group.

2. Let G be a Lie group, $H < G$ a subgroup that is also a regular submanifold.

   (a) Show that H is a Lie group.

   (b) Prove that H is a closed subgroup of G.

3. We consider the determinant function $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^*$. Show that its differential at the identity matrix I is the trace function

$$D_I \det = \mathrm{tr}.$$

4. Let M be a smooth n-dimensional manifold and $p \in M$. Show that if (U, φ) is any chart at p with $\varphi(p) = 0$, then the map

$$\mathbb{R}^n \to T_pM, \quad v \mapsto (f \mapsto D_0(f \circ \varphi^{-1})(v))$$

   is a vector space isomorphism.

5. Let M be a smooth manifold and $p \in M$. Show that for all open sets $U \subseteq M$ with $p \in U$ and every $g \in \mathcal{C}^\infty(U)$, there is $f \in \mathcal{C}^\infty(M)$ such that (U, g) and (M, f) define the same germ at p.

6. We have seen in class that if $\varphi\colon M \to M'$ is a diffeomorphism, then $\mathrm{Vect}^\infty(M) \to \mathrm{Vect}^\infty(M')$, $X \mapsto \varphi_*(X)$ is a Lie algebra isomorphism. Give a formula for $\varphi_*(X)$ in terms of derivatives of φ.