Exercise sheet 5

1. Compute the Lie algebra of $O(p,q)$ and $\mathrm{SO}(p,q)$ for $n = p+q$.

2. Realize $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{C})$ and $U(n)$ as Lie groups, and compute their Lie algebras.

3. Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Show that the Lie algebra of $G \times H$ can be identified with $\mathfrak{g} \times \mathfrak{h}$ with the bracket

$$[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2]_{\mathfrak{g}}, [y_1, y_2]_{\mathfrak{h}}).$$

4. Show that the Lie algebra $(\mathbb{R}^3, \times)$, where $\times$ denotes the cross product, is isomorphic to the Lie algebra of $O(3,\mathbb{R})$.

5. Read and understand the pages from Boothby's book (see website) that give a complete proof of Proposition 3.43 in the notes.